DIRECTORATE OF DISTANCE & CONTINUING EDUCATION

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B.Sc. Physics

II Semester

Vector Calculus and Fourier Series

Unit 1

Vector Functions

If for each value of a scalar variable u, there corresponds a vector f, then f is said to be a vector function of the scalar variable u. The vector function is written as f(u).

Eg., The vector $(acosu)\vec{i}$+$(b\ sinu)\vec{j} + (bu)\vec{k}$ is a vector function of the scalar variable u.

Limit of a vector function

A vector v_0 is said to be the limit of the vector function f(u), if $\lim_{u \to u_0} |f(u) - v_0| = 0$.

i.e., $\lim_{u \to u_0} f(u) = v_0$.

Derivative of a vector function

A vector function $f(u)$ is said to be derivable or differentiable with respect to u, if $\lim_{\Delta u \to 0} \frac{f(u+\Delta u)-f(u)}{\Delta u}$ exists. This limit is called the derivative or differential coefficient of $f(u)$ with respect to u and is denoted by $\frac{df}{du}$.

Note 1 : If $f(u)$ is a constant vector, then its derivative is a zero vector because $f(u + \Delta u) - f(u) = 0$.

Note 2 : If $f(u + \Delta u)$ is written as $f(u) + \Delta f$ then $f(u + \Delta u) - f(u) = \Delta f$ and $\frac{df}{du} = \lim_{\Delta u \to 0} \frac{\Delta f}{\Delta u}$.

Theorem 1 :

(i) If $\emptyset$ is a scalar function of u and 'a' a constant vector, then $\frac{d(\emptyset a)}{du}$=$a\frac{d\emptyset}{du}$

(ii) If 'a' is also a function of u, then $\frac{d(\emptyset a)}{du}$=$\frac{d\emptyset}{du}\vec{a} + \emptyset\frac{da}{du}$.

Proof :

(i) We have $\frac{df}{du} = \lim_{\Delta u \to 0} \frac{\Delta f}{\Delta u}$

Now, $\frac{d(\emptyset a)}{du} = \lim_{\Delta u \to 0} \frac{\Delta(\emptyset a)}{\Delta u}$= $\lim_{\Delta u \to 0} \frac{(\emptyset+\Delta\emptyset)a-\emptyset a}{\Delta u}$= $\lim_{\Delta u \to 0} \frac{[(\emptyset+\Delta\emptyset)-\emptyset]a}{\Delta u}$

$$= \lim_{\Delta u \to 0} \frac{\Delta\emptyset}{\Delta u} a = \frac{d\emptyset}{du} a$$

(ii) Now, $\frac{d(\emptyset a)}{du} = \lim_{\Delta u \to 0} \frac{\Delta(\emptyset a)}{\Delta u}$= $\lim_{\Delta u \to 0} \frac{(\emptyset+\Delta\emptyset)(a+\Delta a)-\emptyset a}{\Delta u}$

$$= \lim_{\Delta u \to 0} \frac{\emptyset a + \emptyset\Delta a + a\Delta\emptyset + \Delta\emptyset \cdot \Delta a - \emptyset a}{\Delta u}$$

$$= \emptyset \lim_{\Delta u \to 0} \frac{\Delta a}{\Delta u} + a \lim_{\Delta u \to 0} \frac{\Delta \emptyset}{\Delta u} + \lim_{\Delta u \to 0} (\frac{\Delta \emptyset}{\Delta u} \cdot \Delta a)$$

$$= \emptyset \lim_{\Delta u \to 0} \frac{\Delta a}{\Delta u} + a \lim_{\Delta u \to 0} \frac{\Delta \emptyset}{\Delta u} + \lim_{\Delta u \to 0} \frac{\Delta \emptyset}{\Delta u} \cdot \lim_{\Delta u \to 0} \Delta a$$

$= \emptyset \lim_{\Delta u \to 0} \frac{\Delta a}{\Delta u} + a \lim_{\Delta u \to 0} \frac{\Delta \emptyset}{\Delta u}$ (since, $\lim_{\Delta u \to 0} \Delta a = 0$)

Thus, $\frac{d(\emptyset a)}{du} = \frac{d\emptyset}{du} \vec{a} + \emptyset \frac{da}{du}$.

Theorem 2 : If A and B are functions of scalar variable u, then prove that $(i) \frac{d(A+B)}{du} = \frac{dA}{du} + \frac{dB}{du}$, $(ii) \frac{d(A \cdot B)}{du} = \frac{dA}{du} B + A \frac{dB}{du}$ and $(iii) \frac{d(A \times B)}{du} = \frac{dA}{du} \times B + A \times \frac{dB}{du}$.

Proof : (i) $\frac{d(A+B)}{du} = \lim_{\Delta u \to 0} \frac{(A+\Delta A)+(B+\Delta B)-(A+B)}{\Delta u}$

$$= \lim_{\Delta u \to 0} \frac{A + \Delta A + B + \Delta B - A - B}{\Delta u} = \lim_{\Delta u \to 0} \frac{\Delta A + \Delta B}{\Delta u} = \lim_{\Delta u \to 0} \frac{\Delta A}{\Delta u} + \lim_{\Delta u \to 0} \frac{\Delta B}{\Delta u}$$

$$= \frac{dA}{du} + \frac{dB}{du}$$

(iii) $\frac{d(A \cdot B)}{du} = \lim_{\Delta u \to 0} \frac{(A+\Delta A) \cdot (B+\Delta B)-(AB)}{\Delta u}$

$$= \lim_{\Delta u \to 0} \frac{AB + A\Delta B + \Delta A \cdot B + \Delta A \cdot \Delta B - (AB)}{\Delta u}$$

$$= \lim_{\Delta u \to 0} A \frac{\Delta B}{\Delta u} + \lim_{\Delta u \to 0} \frac{\Delta A}{\Delta u} B + \lim_{\Delta u \to 0} (\frac{\Delta A}{\Delta u} \Delta B)$$

$= \lim_{\Delta u \to 0} A \frac{\Delta B}{\Delta u} + \lim_{\Delta u \to 0} \frac{\Delta A}{\Delta u} B + \lim_{\Delta u \to 0} \frac{\Delta A}{\Delta u} \lim_{\Delta u \to 0} \Delta B$

$= \lim_{\Delta u \to 0} A \frac{\Delta B}{\Delta u} + \lim_{\Delta u \to 0} \frac{\Delta A}{\Delta u} B$ (Since, $\lim_{\Delta u \to 0} \Delta B = 0$).

$$= A \lim_{\Delta u \to 0} \frac{\Delta B}{\Delta u} + \lim_{\Delta u \to 0} \frac{\Delta A}{\Delta u} B = \frac{dA}{du} B + A \frac{dB}{du}$$

Similarly do the (iii) part.

Problem 1 : Find the derivatives of $\vec{A} \cdot \vec{B}$ and $\vec{A} \times \vec{B}$ with respect to u if $\vec{A} = u^2 \vec{i} + u\vec{j} + 2u\vec{k}$ and $\vec{B} = \vec{j} - u\vec{k}$.

Solution : (i) Find $\frac{d}{du}(\vec{A} \cdot \vec{B})$

$$\vec{A} \cdot \vec{B} = (u^2 \vec{i} + u\vec{j} + 2u\vec{k}) \cdot (\vec{j} - u\vec{k}) = 0 + u - 2u^2 = u - 2u^2$$

$$\frac{d}{du}(\vec{A}\cdot\vec{B}) = \frac{d}{du}(u - 2u^2) = 1 - 4u.$$

(i) Find $\frac{d}{du}(\vec{A}\times\vec{B})$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ u^2 & u & 2u \\ 0 & 1 & -u \end{vmatrix}$$

(Ans.) (-2u-2) $\vec{\imath}$+3u^2 $\vec{\jmath}$+2u$\vec{k}$

Scalar point functions

If for every point P in a domain D of space, there corresponds a scalar $\emptyset$ then $\emptyset$ is said to be a single valued scalar point function defined in the domain D. The value of $\emptyset$ at P is denoted by $\emptyset(P)$ (or) $\emptyset(x,y,z) if\ P\ is\ (x,y,z)$. The function $\emptyset$ is said to be the scalar field in D.

Vector point function

If for every point P in a domain D of space, there corresponds a vector $\emptyset$ then $\emptyset$ is said to be a single valued vector point function defined in the domain D. The value of $\emptyset$ at P is denoted by $\emptyset(P)$ (or) $\emptyset(x,y,z) if\ P\ is\ (x,y,z)$. The function $\emptyset$ is said to be the vector field in D.

Level surfaces

The surfaces represented by the equation $\emptyset = c$ for different values of c are called level surfaces. No two level surfaces will intersect each other.

Directional derivative of a scalar point function

The directional derivative of $\emptyset$ at any point P in the direction specified by the direction cosines l, m, n is $l\frac{\partial\emptyset}{\partial x} + m\frac{\partial\emptyset}{\partial y} + n\frac{\partial\emptyset}{\partial z}$.

Gradient of a scalar point function

If $\emptyset$ *is a* scalar point function, then the vector $\frac{\partial\emptyset}{\partial x}\vec{\imath} + \frac{\partial\emptyset}{\partial y}\vec{\jmath} + \frac{\partial\emptyset}{\partial z}\vec{k}$ is called the gradient of $\emptyset$. This vector is written as grad$\emptyset$ or $\nabla\emptyset$ where ∇ (read as 'del' or 'nebla') stands for $\vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$.

Note 1

The operator ∇ is an operator whose function is to transform a scalar point function $\emptyset$ into a vector point function.

Note 2

The summation notation for gradient is $\nabla\emptyset = \sum \vec{i}\frac{\partial\emptyset}{\partial x}$.

The directional derivative of ∅ in the direction specified by the unit vector $\vec{e}$ *is* $\nabla\emptyset \cdot \vec{e}$.

Let the direction cosines of $\vec{e}$ is l, m, n. Then $\vec{e} = l\vec{i} + m\vec{j} + n\vec{k}$.

Now, $\nabla\emptyset \cdot \vec{e} = \left(\frac{\partial\emptyset}{\partial x}\vec{i} + \frac{\partial\emptyset}{\partial y}\vec{j} + \frac{\partial\emptyset}{\partial z}\vec{k}\right) \cdot (l\vec{i} + m\vec{j} + n\vec{k}) = l\frac{\partial\emptyset}{\partial x} + m\frac{\partial\emptyset}{\partial y} + n\frac{\partial\emptyset}{\partial z}$

which is the directional derivative of ∅ in the direction whose direction cosines are l, m, n.

Note : Maximum value of the directional derivative of ∅ is $|\nabla\emptyset|$.

Theorem

If ∅ and ψ are scalar point functions, then prove that

(i) $\nabla(k\emptyset) = k(\nabla\emptyset)$ ***where k is a constant***

(ii) $\nabla(\emptyset + \psi) = \nabla\emptyset + \nabla\psi$

(iii) $\nabla(\emptyset\psi) = (\nabla\emptyset)\psi + \emptyset(\nabla\psi)$

(iv) $\nabla\left(\frac{\emptyset}{\psi}\right) = \frac{\psi(\nabla\emptyset) - \emptyset(\nabla\psi)}{\psi^2}$.

Proof : (i) $\nabla(k\emptyset) = \sum \vec{i}\frac{\partial(k\emptyset)}{\partial x} = \sum \vec{i}k\frac{\partial\emptyset}{\partial x} = k\sum \vec{i}\frac{\partial\emptyset}{\partial x} = k(\nabla\emptyset)$

(ii) As in proof (i)

(iii) $\nabla(\emptyset\psi) = \sum \vec{i}\frac{\partial(\emptyset\psi)}{\partial x} = \sum \vec{i}\left(\frac{\partial\emptyset}{\partial x}\psi + \emptyset\frac{\partial\psi}{\partial x}\right) = \sum \vec{i}\frac{\partial\emptyset}{\partial x}\psi + \sum \vec{i}\emptyset\frac{\partial\psi}{\partial x}$

$= \psi\sum \vec{i}\frac{\partial\emptyset}{\partial x} + \emptyset\sum \vec{i}\frac{\partial\psi}{\partial x} = (\nabla\emptyset)\psi + \emptyset(\nabla\psi)$

(iv) As in proof (iii).

Problem 2 : ***Find the directional derivative of $x + xy^2 + yz^3$ at the point $(0,1,1)$ in the direction whose d.c's are 2/3, 2/3, -1/3***

Soln : Let $\emptyset = x + xy^2 + yz^3$

Find $\frac{\partial\emptyset}{\partial x}, \frac{\partial\emptyset}{\partial y}, \frac{\partial\emptyset}{\partial z}$

Given $l = 2/3, m = 2/3$ *and* $n = -1/3$

The directional derivative is $l\frac{\partial\emptyset}{\partial x} + m\frac{\partial\emptyset}{\partial y} + n\frac{\partial\emptyset}{\partial z}$

$= \frac{2}{3}(1 + y^2) + \frac{2}{3}(2xy + z^3) - yz^2$.

At the point (0,1,1) [Ans. 1]

Problem 3. Find $\nabla\emptyset$ at (x,y,z) if $\emptyset = x + xy^2 + yz^3$

Problem 4: Find the directional derivative of $\emptyset = 3xy^2 - x^2yz$ at the point (1,2,3) in the direction of the vector $\vec{i} - 2\vec{j} + 2\vec{k}$.

{Hint : Find $\nabla\emptyset$ then find $\vec{e}$ as $\frac{1}{3}\vec{i} - \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}$. then find $\nabla\emptyset \cdot \vec{e}$. Ans. : -22/3}

Problem 5: Find the directional derivative of $\emptyset = x^3 + y^3 + z^3$ at the point (1,-1,2) in the direction of the vector $\vec{i} + 2\vec{j} + \vec{k}$. {Ans. $\frac{21}{\sqrt{6}}$}

Problem 6: ***If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ (i.e.,) if $\vec{r}$ is the position vector of the variable point (x,y,z) and $|\vec{r}| = r$. Show that (i) $\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$ and (ii) $\nabla\big(f(r)\big) = f'(r)\hat{r}$.***

Proof : Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$ (i.e.,) $r^2 = x^2 + y^2 + z^2$

Differentiating partially with respect to x. $2r\frac{\partial r}{\partial x} = 2x$. $\therefore \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ *and* $\frac{\partial r}{\partial z} = \frac{z}{r}$

Proof of (i) $\nabla\left(\frac{1}{r}\right) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right) = \vec{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \vec{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \vec{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$

$$= \vec{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + \vec{j}\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right) + \vec{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= -\frac{1}{r^2}\vec{i}\left(\frac{\partial r}{\partial x}\right) - \frac{1}{r^2}\vec{j}\left(\frac{\partial r}{\partial y}\right) - \frac{1}{r^2}\vec{k}\left(\frac{\partial r}{\partial z}\right) = -\frac{1}{r^2}(\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r})$$

$$= -\frac{1}{r^3}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = -\frac{\vec{r}}{r^3}$$

(ii) T.P. $\nabla\big(f(r)\big) = f'(r)\hat{r}$.

$\nabla\big(f(r)\big) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(f(r)) = \vec{i}\frac{\partial}{\partial x}(f(r)) + \vec{j}\frac{\partial}{\partial y}(f(r)) + \vec{k}\frac{\partial}{\partial z}(f(r))$ (complete the problem) {Hint $\frac{\vec{r}}{r} = \hat{r}$}.

Problem 7 : ***If $\nabla\emptyset = 5r^3\vec{r}$ then find $\emptyset$.***

Solution : We have, $\frac{\vec{r}}{r} = \hat{r} => \vec{r} = r\hat{r}$

$$\therefore \nabla\emptyset = 5r^3r\hat{r} = 5r^4\hat{r}$$

We have, $\nabla\emptyset = \emptyset'(r)\hat{r}$

$$\therefore \nabla\emptyset = \emptyset'(r)\hat{r} = 5r^4\hat{r}$$

$$\therefore \emptyset'(r) = 5r^4$$

Integrating with respect to r $\quad \int \emptyset'(r)dr = \int 5r^4 dr$

$\emptyset(r) = \frac{5r^5}{5} + c$ i.e., $\emptyset(r) = r^5 + c.$

Problem 8:* *If $\boldsymbol{\nabla\emptyset = (6r - 3r^2)\vec{r}}$ ***and*** $\boldsymbol{\emptyset(2) = 4}$ ***then find*** $\boldsymbol{\emptyset}$.

{Hint : Find the value of c using the condition $\emptyset(2) = 4$. (Ans. $\emptyset(r) = 2(r^3 - r^4 + 10)$.

Problem 9 *: If* $\boldsymbol{\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}}$ ***(i.e.,) if*** $\boldsymbol{\vec{r}}$ ***is the position vector of the variable point (x,y,z) and*** $\boldsymbol{|\vec{r}| = r}$***, then show that***

(i) $\quad \boldsymbol{\nabla(\log r) = \frac{\vec{r}}{r^2}}$

(ii) $\quad \boldsymbol{\nabla r^n = nr^{n-1}\hat{r} = nr^{n-2}\vec{r}}$

(iii) $\quad \boldsymbol{\nabla(\vec{r}\cdot\vec{a}) = \vec{a}}$ ***where a is a constant vector.***

(iv) $\quad \boldsymbol{\nabla(\vec{a}\cdot\vec{r}) = 2a}$ ***if*** $\boldsymbol{\vec{a} = \alpha x\vec{\imath} + \beta y\vec{\jmath} + \gamma z\vec{k}}$.

Proof : Given $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$.

$|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$ (i.e.,) $r^2 = x^2 + y^2 + z^2$

Differentiating partially with respect to x. $2r\frac{\partial r}{\partial x} = 2x. \quad \therefore \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

(i) $\quad \nabla(\log r) = \left(\vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\log r)$

$$= \vec{\imath}\frac{\partial}{\partial x}(\log r) + \vec{\jmath}\frac{\partial}{\partial y}(\log r) + \vec{k}\frac{\partial}{\partial z}(\log r)$$

$$= \vec{\imath}\left(\frac{1}{r}\frac{\partial r}{\partial x}\right) + \vec{\jmath}\left(\frac{1}{r}\frac{\partial r}{\partial y}\right) + \vec{k}\left(\frac{1}{r}\frac{\partial r}{\partial z}\right)$$

$$= \frac{1}{r}\vec{\imath}\left(\frac{\partial r}{\partial x}\right) + \frac{1}{r}\vec{\jmath}\left(\frac{\partial r}{\partial y}\right) + \frac{1}{r}\vec{k}\left(\frac{\partial r}{\partial z}\right) = \frac{1}{r}(\vec{\imath}\frac{x}{r} + \vec{\jmath}\frac{y}{r} + \vec{k}\frac{z}{r})$$

$$= \frac{1}{r^2}\left(x\vec{\imath} + y\vec{\jmath} + z\vec{k}\right) = \frac{\vec{r}}{r^2}$$

(ii) $\quad$ As in the part (i)

(iii) $\quad$ T.P. $\nabla(\vec{r}\cdot\vec{a}) = \vec{a}$

Given a is a constant vector. $\therefore \vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\vec{r}\cdot\vec{a} = (x\vec{i} + y\vec{j} + z\vec{k})\cdot(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) = a_1x + a_2y + a_3z$

$$\nabla(\vec{r}\cdot\vec{a}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$$

$$= \vec{i}\frac{\partial}{\partial x}(a_1x + a_2y + a_3z) + \vec{j}\frac{\partial}{\partial y}(a_1x + a_2y + a_3z) + \vec{k}\frac{\partial}{\partial z}(a_1x + a_2y + a_3z)$$

$$= a_1\vec{i} + a_2\vec{j} + a_3\vec{k} = \vec{a}$$

(iv) As in the part (iii)

Problem 10: If $\nabla\emptyset = (y + y^2 + z^2)\vec{i} + (x + z + 2xy)\vec{j} + (y + 2zx)\vec{k}$ and if $\emptyset(1,1,1) = 3, find\ \emptyset$.

Solution : Given, $\nabla\emptyset = (y + y^2 + z^2)\vec{i} + (x + z + 2xy)\vec{j} + (y + 2zx)\vec{k}$ ..... (1)

We have, $\nabla\emptyset = \frac{\partial\emptyset}{\partial x}\vec{i} + \frac{\partial\emptyset}{\partial y}\vec{j} + \frac{\partial\emptyset}{\partial z}\vec{k}$ ....(2)

From (1) and (2) we have $\frac{\partial\emptyset}{\partial x} = (y + y^2 + z^2)$ ......(3)

$\frac{\partial\emptyset}{\partial y} = (x + z + 2xy)$ .....(4) & $\frac{\partial\emptyset}{\partial z} = y + 2zx$ ......(5)

Integrating (3) w.r.to x, $\emptyset = yx + y^2x + xz^2 + f(y,z)$ .... (6)

Integrating (4) w.r. to y, $\emptyset = xy + zy + xy^2 + g(x,z)$ ...... (7)

Integrating (5) w.r.to z, $\emptyset = yz + z^2x + h(x,y)$....... (8)

From (6), (7) and (8) we get, $\emptyset = yx + y^2x + xz^2 + yz + c$

$Given, \emptyset(1,1,1) = 3$. Therefore, 1+1+1+1+c=3 => $c = -1$

Hence, $\emptyset = yx + y^2x + xz^2 + yz - 1$.

Problem 11 : Find $\emptyset$ if $\nabla\emptyset$ is $(6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$

(Ans. : $3x^2y + xz^3 - yz + c$)

Problem 12 : Find the unit vectors normal to the following surfaces.

(i) $x^2 + 2y^2 + z^2 = 7\ at\ (1,-1,2)$

(ii) $x^2 + y^2 - z^2 = 1\ at\ (1,1,1)$ [Ans. $\frac{\vec{i}+\vec{j}-\vec{k}}{\sqrt{3}}$]

(iii) $x^2 + 3y^2 + 2z^2 = 6\ at\ (2,0,1)$ [Ans. $\frac{\vec{i}+\vec{k}}{\sqrt{2}}$]

Solution : (i)Let $\emptyset = x^2 + 2y^2 + z^2 - 7$

$$\nabla\emptyset = 2x\vec{i} + 4y\vec{j} + 2z\vec{k}$$

At $(1,-1,2), \nabla\emptyset = 2\vec{i} - 4\vec{j} + 4\vec{k}$

$$|\nabla\emptyset| = 6\ (Verify)$$

Unit vector normal to the surface = $\frac{\nabla\emptyset}{|\nabla\emptyset|} = \frac{\vec{i}-2\vec{j}+2\vec{k}}{3}$

(ii) & (iii) As in 1st part.

Problem 12 : Find the equation of the tangent plane to the surface $x^2 + 2y^2 + 3z^2 = 6$ at the point (1,-1,1).

Sol. : Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Let $\emptyset = x^2 + 2y^2 + 3z^2 - 6$

$$\nabla\emptyset = 2x\vec{i} + 4y\vec{j} + 6z\vec{k}$$

At $(1,-1,1), \nabla\emptyset = 2\vec{i} - 4\vec{j} + 6\vec{k} = \vec{P}$

Let $\vec{r_1} = \vec{i} - \vec{j} + \vec{k}$

Equation of the tangent plane is $(\vec{r} - \vec{r_1}) \cdot \vec{P} = 0$

x-2y+3z-6=0 (Verify)

Problem 13 : Find the equation of the tangent plane to the surface $x^2 - 4y^2 + 3z^2 + 4 = 0$ at the point (3,2,1). [Ans. 3x-8y+3z+4=0]

Problem 14 : Find the angle between the normals to the surface $xy - z^2 = 0$ at the points (1,4,-2) and (-3,-3,3)

Solution : First find $\nabla\emptyset$ at the points (1,4,-2) and (-3,-3,3).

At $(1,4,-2), \nabla\emptyset = 4\vec{i} + \vec{j} + 4\vec{k}$ & At $(-3,-3,3), \nabla\emptyset = -3\vec{i} - 3\vec{j} - 6\vec{k}$

If θ is the angle between the normals then $cos\theta = \frac{(4\vec{i}+\vec{j}+4\vec{k})\cdot(-3\vec{i}-3\vec{j}-6\vec{k})}{\sqrt{16+1+16}\sqrt{9+9+36}} = \frac{-13}{3\sqrt{22}}$

Problem 15: Show that the surfaces $5x^2 - 2yz - 9x = 0\ and\ 4x^2y + z^3 - 4 = 0$ are orthogonal at (1,-1,2).

Soln. : - Let $\emptyset_1 = 5x^2 - 2yz - 9x$ and $\emptyset_2 = 4x^2y + z^3 - 4$

$$\nabla\emptyset_1 = (10x - 9)\vec{i} - 2z\vec{j} - 2y\vec{k}\ \&\ \nabla\emptyset_2 = 8xy\vec{i} + 4x^2\vec{j} + 3z^2\vec{k}\ (verify)$$

At (1,-1,2) $\nabla\emptyset_1 = \vec{i} - 4\vec{j} + 2\vec{k}$ & $\nabla\emptyset_2 = -8\vec{i} + 4\vec{j} + 12\vec{k}\ (verify)$

$$(\nabla\emptyset_1)\cdot(\nabla\emptyset_2) = (\vec{i} - 4\vec{j} + 2\vec{k}) \cdot (-8\vec{i} + 4\vec{j} + 12\vec{k}) = 0\ (verify)$$

Thus the given two surfaces are orthogonal.

Problem 16: Find the angle between the normals to the intersecting surfaces $xy - z^2 - 1 = 0$ and $y^2 - 3z - 1 = 0$ at (1,1,0). Also find a unit vector along the tangent to the curve of intersection of the surfaces at (1,1,0).

Soln. : As in the previous problem find $\nabla\emptyset_1$ & $\nabla\emptyset_2\ at$ (1,1,0)

$$\nabla\emptyset_1 = \vec{i} + \vec{j}\ \&\ \nabla\emptyset_2 = 2\vec{j} - 3\vec{k}$$

Let $\vec{a} = \vec{i} + \vec{j}\ \&\ \vec{b} = 2\vec{j} - 3\vec{k}$

Let θ bet the angle between the normals to the surfaces.

$$\therefore cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{2}{\sqrt{26}}$$

Unit Vector along the tangent = $\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 0 & 2 & -3 \end{vmatrix} = -3\vec{i} + 3\vec{j} + 2\vec{k}$ and $|\vec{a}\times\vec{b}| = \sqrt{22}$ (Verify)

Thus the unit vector along the tangent = $\frac{-3\vec{i}+3\vec{j}+2\vec{k}}{\sqrt{22}}$.

Problem 17: Find the direction in which $\emptyset = xy^2 + yz^2 + zx^2$ increases most rapidly at the point (1,2,-3).

Soln. : Find $\nabla\emptyset$ at (1,2,-3)

Direction of $\nabla(xy^2 + yz^2 + zx^2) = -2\vec{i} + 13\vec{j} - 11\vec{k}$.

Divergence of a vector point function

If $V = V_1\vec{i} + V_2\vec{j} + V_3\vec{k}$ is a vector point function, then the scalar $\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$ is called the **divergence of V** and is denoted by divV (or) $\nabla\cdot V$.

If $\nabla\cdot V = 0$ then the vector V is said to be **solenoidal**.

The summation notation for divergence is $\nabla\cdot V = \sum\vec{i}\cdot\frac{\partial V}{\partial x}$.

Curl of a vector point function

If $\boldsymbol{V} = V_1\vec{\imath} + V_2\vec{\jmath} + V_3\vec{k}$, then the vector $\vec{\imath}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \vec{\jmath}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \vec{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$ is called the **curl of V** and is denoted by **curlV** (or) $\nabla \times V$.

$$\text{Now, } \nabla \times V = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

If $\boldsymbol{\nabla} \times \boldsymbol{V} = \boldsymbol{0}$ then the vector **V** is said to be **irrotational**.

Note 1 : The divergence of a vector point function is a scalar and the curl of a vector point function is a vector.

Note 2 : $V \cdot \nabla = V \cdot \sum \vec{\imath}\frac{\partial}{\partial x}$

Theorem 1 : If $\vec{A}$ and $\vec{B}$ are vector point functions, $'\emptyset'$ a scalar point function and ‘k’ a constant then, (i) $\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$ (ii) $\nabla \cdot (k\vec{A}) = k(\nabla \cdot \vec{A})$

(iii) $\nabla \cdot (\emptyset\vec{A}) = (\nabla\emptyset) \cdot \vec{A} + \emptyset(\nabla \cdot \vec{A})$ (iv) $\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$

(v) $\nabla \times (k\vec{A}) = k(\nabla \times \vec{A})$ (vi) $\nabla \times (\emptyset\vec{A}) = (\nabla\emptyset) \times \vec{A} + \emptyset(\nabla \times \vec{A})$

Proof : (i) $\nabla \cdot (\vec{A} + \vec{B}) = \sum \vec{\imath} \cdot \frac{\partial(\vec{A}+\vec{B})}{\partial x} = \sum \vec{\imath} \cdot (\frac{\partial \vec{A}}{\partial x} + \frac{\partial \vec{B}}{\partial x}) = \sum \vec{\imath} \cdot \frac{\partial \vec{A}}{\partial x} + \sum \vec{\imath} \cdot \frac{\partial \vec{B}}{\partial x}$

$= \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$

(ii) $\nabla \cdot (k\vec{A}) = \sum \vec{\imath} \cdot \frac{\partial(k\vec{A})}{\partial x} = k \sum \vec{\imath} \cdot \frac{\partial \vec{A}}{\partial x}$ [Since, k is a constant]$= k(\nabla \cdot \vec{A})$

(iii) $\nabla \cdot (\emptyset\vec{A}) = \sum \vec{\imath} \cdot \frac{\partial(\emptyset\vec{A})}{\partial x} = \sum \vec{\imath} \cdot [\frac{\partial \emptyset}{\partial x}\vec{A} + \emptyset\frac{\partial \vec{A}}{\partial x}] = \sum \vec{\imath} \cdot \frac{\partial \emptyset}{\partial x}\vec{A} + \sum \vec{\imath} \cdot \emptyset\frac{\partial \vec{A}}{\partial x}$

$= \sum \vec{\imath}\frac{\partial \emptyset}{\partial x} \cdot \vec{A} + \emptyset \sum \vec{\imath} \cdot \frac{\partial \vec{A}}{\partial x} = (\nabla\emptyset) \cdot \vec{A} + \emptyset(\nabla \cdot \vec{A})$

(iv), (v) & (vi) are H.W.

Theorem 2 : If $\vec{A}$ and $\vec{B}$ are vector point functions then,

(i) $\nabla(\vec{A} \cdot \vec{B}) = \vec{A} \times (\nabla \times \vec{B}) + (\vec{A} \cdot \nabla)\vec{B} + \vec{B} \times (\nabla \times \vec{A}) + (\vec{B} \cdot \nabla)\vec{A}$

(ii) $\nabla \cdot (\vec{A} \times \vec{B}) = (\nabla \times \vec{A}) \cdot \vec{B} - (\nabla \times \vec{B}) \cdot \vec{A}$

(iii) $\nabla \times (\vec{A} \times \vec{B}) = \{(\vec{B} \cdot \nabla)\vec{A} - (\nabla \cdot \vec{A})\vec{B}\} - \{(\vec{A} \cdot \nabla)\vec{B} - (\nabla \cdot \vec{B})\vec{A}\}$

Proof : (i) $\nabla(\vec{A} \cdot \vec{B}) = \sum \vec{\imath} \cdot \frac{\partial(\vec{A} \cdot \vec{B})}{\partial x} = \sum \vec{\imath}\left[\frac{\partial \vec{A}}{\partial x} \cdot \vec{B} + \vec{A} \cdot \frac{\partial \vec{B}}{\partial x}\right] = \sum \vec{\imath}\left[\frac{\partial \vec{A}}{\partial x} \cdot \vec{B}\right] + \sum \vec{\imath}\left[\vec{A} \cdot \frac{\partial \vec{B}}{\partial x}\right] = \sum \vec{\imath}\left[\vec{B} \cdot \frac{\partial \vec{A}}{\partial x}\right] + \sum \vec{\imath}\left[\vec{A} \cdot \frac{\partial \vec{B}}{\partial x}\right]$ .... (1)

Now, $\vec{A}\times\left(\vec{\imath}\times\frac{\partial\vec{B}}{\partial x}\right)=\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\vec{\imath}-(\vec{A}\cdot\vec{\imath})\frac{\partial\vec{B}}{\partial x}$

(Using, $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$)

$\vec{A}\times\left(\vec{\imath}\times\frac{\partial\vec{B}}{\partial x}\right)+(\vec{A}\cdot\vec{\imath})\frac{\partial\vec{B}}{\partial x}=\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\vec{\imath}$

Taking summation on both sides.

$$\sum[\vec{A}\times\left(\vec{\imath}\times\frac{\partial\vec{B}}{\partial x}\right)+(\vec{A}\cdot\vec{\imath})\frac{\partial\vec{B}}{\partial x}]=\sum[\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\vec{\imath}]$$

$$\sum[\vec{A}\times\left(\vec{\imath}\times\frac{\partial\vec{B}}{\partial x}\right)]+\sum[(\vec{A}\cdot\vec{\imath})\frac{\partial\vec{B}}{\partial x}]=\sum[\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\vec{\imath}]$$

$$\sum[\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\vec{\imath}]=\sum[\vec{A}\times\left(\vec{\imath}\times\frac{\partial\vec{B}}{\partial x}\right)]+\sum[(\vec{A}\cdot\vec{\imath})\frac{\partial\vec{B}}{\partial x}]$$

$$=\vec{A}\times\sum\left(\vec{\imath}\times\frac{\partial\vec{B}}{\partial x}\right)+[\vec{A}\cdot\sum\vec{\imath}\frac{\partial}{\partial x}]\vec{B}$$

$=\vec{A}\times(\nabla\times\vec{B})+(\vec{A}\cdot\nabla)\vec{B}$ .... (2)

Interchanging $\vec{A}$ and $\vec{B}$, we get

$\sum[\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right)\vec{\imath}]=\vec{B}\times(\nabla\times\vec{A})+(\vec{B}\cdot\nabla)\vec{A}$ .... (3)

Put (2) and (3) in (1) we get,

$$\nabla(\vec{A}\cdot\vec{B})=\vec{A}\times(\nabla\times\vec{B})+(\vec{A}\cdot\nabla)\vec{B}+\vec{B}\times(\nabla\times\vec{A})+(\vec{B}\cdot\nabla)\vec{A}$$

(ii) $\nabla\cdot(\vec{A}\times\vec{B})=\sum\vec{\imath}\cdot\frac{\partial(\vec{A}\times\vec{B})}{\partial x}=\sum\vec{\imath}\cdot\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}+\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)$

$$=\sum\vec{\imath}\cdot(\frac{\partial\vec{A}}{\partial x}\times\vec{B})+\sum\vec{\imath}\cdot(\vec{A}\times\frac{\partial\vec{B}}{\partial x})=\sum\vec{\imath}\cdot\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)-\sum\vec{\imath}\cdot(\frac{\partial\vec{B}}{\partial x}\times\vec{A})$$

$=\sum\vec{\imath}\times\left(\frac{\partial\vec{A}}{\partial x}\cdot\vec{B}\right)-\sum\vec{\imath}\times(\frac{\partial\vec{B}}{\partial x}\cdot\vec{A})$ [Interchanging dot and cross]

$$=\sum\left(\vec{\imath}\times\frac{\partial\vec{A}}{\partial x}\right)\cdot\vec{B}-\sum(\vec{\imath}\times\frac{\partial\vec{B}}{\partial x})\cdot\vec{A}=(\nabla\times\vec{A})\cdot\vec{B}-(\nabla\times\vec{B})\cdot\vec{A}$$

(iii) $\nabla\times(\vec{A}\times\vec{B})=\sum\vec{\imath}\times\frac{\partial(\vec{A}\times\vec{B})}{\partial x}=\sum\vec{\imath}\times\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}+\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)$

$=\sum\vec{\imath}\times\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)+\sum\vec{\imath}\times\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)$

$$= \sum \vec{i} \times \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B}\right) - \sum \vec{i} \times (\frac{\partial \vec{B}}{\partial x} \times \vec{A}) \text{ .... (1)}$$

Now, $\vec{i} \times \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B}\right) = (\vec{i} \cdot \vec{B})\frac{\partial \vec{A}}{\partial x} - (\vec{i} \cdot \frac{\partial \vec{A}}{\partial x})\vec{B}$

$$\sum \vec{i} \times \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B}\right) = \sum(\vec{i} \cdot \vec{B})\frac{\partial \vec{A}}{\partial x} - \sum(\vec{i} \cdot \frac{\partial \vec{A}}{\partial x})\vec{B}$$

$$= \sum(\vec{B} \cdot \vec{i})\frac{\partial \vec{A}}{\partial x} - \sum\left(\vec{i} \cdot \frac{\partial \vec{A}}{\partial x}\right)\vec{B} = \vec{B} \cdot \sum \vec{i}\frac{\partial \vec{A}}{\partial x} - \sum\left(\vec{i} \cdot \frac{\partial \vec{A}}{\partial x}\right)\vec{B}$$

$$= (\vec{B} \cdot \sum \vec{i}\frac{\partial}{\partial x})\ \vec{A} - \sum\left(\vec{i} \cdot \frac{\partial \vec{A}}{\partial x}\right)\vec{B}$$

$$= (\vec{B} \cdot \nabla)\vec{A} - (\nabla \cdot \vec{A})\vec{B} \text{ .... (2)}$$

Interchanging $\vec{A}$ & $\vec{B}$ we get, $\sum \vec{i} \times \left(\frac{\partial \vec{B}}{\partial x} \times \vec{A}\right) = (\vec{A} \cdot \nabla)\vec{B} - (\nabla \cdot \vec{B})\vec{A}$ .... (3)

Put (2) and (3) in (1) we get the result.

Laplacian Differential operator

The operator ∇^2 defined by $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called Laplacian differential operator.

Laplace Equation

If $\emptyset$ is such that $\boldsymbol{\nabla^2\emptyset = 0}$ $(i.e., \frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} + \frac{\partial^2\emptyset}{\partial z^2} = 0)$ the $\emptyset$ is said to satisfied Laplace equation.

Harmonic function

A single valued function f(x,y,z) is said to be a harmonic function if its second order partial derivative exists and are continuous and if the function satisfies the Laplace equation $\boldsymbol{\nabla^2 f = 0}$.

Theorem 3 : If $\emptyset$ is a scalar point function then,

(i) Divergence of the gradient of $\emptyset$ is $\nabla^2\emptyset$ $i.e., \nabla \cdot (\nabla\emptyset) = \nabla^2\emptyset$

(ii) Curl of the gradient of $\emptyset$ vanishes. (i.e.,) $\nabla \times (\nabla\emptyset) = \vec{0}$.

Proof :

(i) We have, $\nabla\emptyset = \vec{i}\frac{\partial\emptyset}{\partial x} + \vec{j}\frac{\partial\emptyset}{\partial y} + \vec{k}\frac{\partial\emptyset}{\partial z}$

$$\nabla \cdot \nabla\emptyset = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot (\vec{i}\frac{\partial\emptyset}{\partial x} + \vec{j}\frac{\partial\emptyset}{\partial y} + \vec{k}\frac{\partial\emptyset}{\partial z})$$

$$= \frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} + \frac{\partial^2 \emptyset}{\partial z^2} = \nabla^2 \emptyset$$

(ii) $\nabla \times (\nabla \emptyset) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \emptyset}{\partial x} & \frac{\partial \emptyset}{\partial y} & \frac{\partial \emptyset}{\partial z} \end{vmatrix} = 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}$ (Verify)

Theorem 4 : If $\vec{A} = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$ where A_1, A_2, A_3 have continuous second partials, then

(i) divergence of a curl of a vector vanishes. i.e., $\nabla \cdot (\nabla \times \vec{A}) = 0$

(ii) $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

Proof : (i) $\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$

$$= \vec{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \vec{j}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \vec{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$\nabla \cdot (\nabla \times \vec{A}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left[\vec{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \vec{j}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \vec{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\right]$$

$$= \frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial x \partial z} - \frac{\partial^2 A_3}{\partial x \partial y} + \frac{\partial^2 A_1}{\partial y \partial z} + \frac{\partial^2 A_2}{\partial x \partial z} - \frac{\partial^2 A_1}{\partial y \partial z} = 0$$

ii) $\nabla \times \vec{A} = \vec{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \vec{j}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \vec{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$

$$\nabla \times (\nabla \times \vec{A}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} & -\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) & \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right)\right] - \vec{j} \ldots\ldots..$$

$$= \sum \vec{i}\left[\frac{\partial^2 A_2}{\partial x \partial y} - \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_3}{\partial x \partial z} - \frac{\partial^2 A_1}{\partial z^2}\right]$$

$$= \sum \vec{i}\left[\frac{\partial^2 A_2}{\partial x \partial y} + \frac{\partial^2 A_3}{\partial x \partial z} - \left(\frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2}\right)\right]$$

$$= \sum \vec{i}\left[\frac{\partial^2 A_2}{\partial x \partial y} + \frac{\partial^2 A_3}{\partial x \partial z} + \frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 A_1}{\partial x^2} - \left(\frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2}\right)\right]$$

(Add and subtract $\frac{\partial^2 A_1}{\partial x^2}$)

$$= \sum \vec{i}\left[\frac{\partial^2 A_2}{\partial x \partial y} + \frac{\partial^2 A_3}{\partial x \partial z} + \frac{\partial^2 A_1}{\partial x^2} - \left(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2}\right)\right]$$

$$= \sum \vec{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} + \frac{\partial A_1}{\partial x}\right) - \left(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2}\right)\right]$$

$$= \sum \vec{i}\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) - \sum \vec{i}\left(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2}\right)$$

$= \sum \vec{i}\frac{\partial}{\partial x}(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}.$

Problem 18 : Show that the vector $\vec{A} = x^2 z^2 \vec{i} + xyz^2 \vec{j} - xz^3 \vec{k}$ is solenoidal.

Solution : (To show that $\nabla \cdot \vec{A}=0$)

Problem 19: If the vector $3x\vec{i} + (x+y)\vec{j} - az\vec{k}$ *is solenoidal*, find a.

Solution : Let $\vec{A} = 3x\vec{i} + (x+y)\vec{j} - az\vec{k}$

Given $\vec{A}$ is solenoidal. Therefore, $\nabla \cdot \vec{A}=0$

i.e., $\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(3x\vec{i} + (x+y)\vec{j} - az\vec{k}\right) = 0$

i.e., $\frac{\partial(3x)}{\partial x} + \frac{\partial(x+y)}{\partial y} - \frac{\partial(az)}{\partial z} = 0$

i.e., 3+1-a=0 =>a=4.

Problem 20 : If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, Show that $\nabla \cdot \vec{r} = 3$.

Solution : Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\nabla \cdot \vec{r} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

$$= \frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(z)}{\partial z} = 1 + 1 + 1 = 3$$

Problem 21 : If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ *and* $r = |\vec{r}|$ show that $\nabla \cdot (r^n \vec{r}) = (n+3)r^n$.

Solution : $\nabla \cdot (r^n \vec{r}) = \nabla(r^n) \cdot \vec{r} + r^n(\nabla \cdot \vec{r})$ .... (1)

$$\nabla(r^n) = \vec{i}\frac{\partial(r^n)}{\partial x} + \vec{j}\frac{\partial(r^n)}{\partial y} + \vec{k}\frac{\partial(r^n)}{\partial z}$$

$$= \vec{i}nr^{n-1}\frac{\partial r}{\partial x} + \vec{j}nr^{n-1}\frac{\partial r}{\partial y} + \vec{k}nr^{n-1}\frac{\partial r}{\partial z}$$

$$= \vec{i}nr^{n-1}\frac{x}{r} + \vec{j}nr^{n-1}\frac{y}{r} + \vec{k}nr^{n-1}\frac{z}{r}$$

$$= nr^{n-2}(\vec{i}x + \vec{j}y + \vec{k}z) = nr^{n-2}\vec{r}$$

$\therefore$ (1) => $\nabla \cdot (r^n\vec{r}) = (nr^{n-2}\vec{r}\cdot\vec{r}) + r^n 3 = nr^{n-2}(\vec{r}\cdot\vec{r}) + r^n 3$

$$= nr^{n-2}r^2 + r^n 3 = nr^n + r^n 3 = (n+3)r^n$$

Problem 22 : Show that $\nabla \cdot \left(\frac{1}{r^3}\vec{r}\right) = 0$ & $\nabla \cdot \hat{r} = \frac{2}{r}$

Solution : $\nabla \cdot \left(\frac{1}{r^3}\vec{r}\right) = \nabla(\frac{1}{r^3}) \cdot \vec{r} + \frac{1}{r^3}(\nabla \cdot \vec{r})$ .... (1)

$$\nabla\left(\frac{1}{r^3}\right) = \vec{i}\frac{\partial\left(\frac{1}{r^3}\right)}{\partial x} + \vec{j}\frac{\partial\left(\frac{1}{r^3}\right)}{\partial y} + \vec{k}\frac{\partial\left(\frac{1}{r^3}\right)}{\partial z}$$

$$= \vec{i}\left(-\frac{3}{r^4}\right)\frac{\partial \mathrm{r}}{\partial x} + \vec{j}\left(-\frac{3}{r^4}\right)\frac{\partial \mathrm{r}}{\partial y} + \vec{k}\left(-\frac{3}{r^4}\right)\frac{\partial \mathrm{r}}{\partial z}$$

$$= \vec{i}\left(-\frac{3}{r^4}\right)\frac{x}{r} + \vec{j}\left(-\frac{3}{r^4}\right)\frac{y}{r} + \vec{k}\left(-\frac{3}{r^4}\right)\frac{z}{r}$$

$$= \left(-\frac{3}{r^5}\right)(\vec{i}x + \vec{j}y + \vec{k}z) = \left(-\frac{3}{r^5}\right)\vec{r}$$

$\therefore$ (1) => $\nabla \cdot \left(\frac{1}{r^3}\vec{r}\right) = \left(-\frac{3}{r^5}\right)\vec{r}\cdot\vec{r} + \frac{1}{r^3}3 = \left(-\frac{3}{r^3}\right) + \frac{1}{r^3}3 = 0$

$$\nabla \cdot \hat{r} = \nabla \cdot \frac{\vec{r}}{r} = \nabla \cdot \frac{1}{r}\vec{r}$$

$$\nabla \cdot \left(\frac{1}{r}\vec{r}\right) = \nabla\left(\frac{1}{r}\right)\cdot\vec{r} + \frac{1}{r}(\nabla\cdot\vec{r})$$

$$\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^3}\vec{r}\ (Verify)$$

$$\nabla \cdot \left(\frac{1}{r}\vec{r}\right) = \left(-\frac{1}{r^3}\vec{r}\right)\cdot\vec{r} + \frac{1}{r}3 = -\frac{1}{r^3}\times r^2 + \frac{3}{r} = \frac{2}{r}$$

Problem 23 : If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}\ and\ r = |\vec{r}|$ show that

$\nabla \cdot (f(r)\vec{r}) = r\,f'(r) + 3\,f(r)$. Also if $\nabla \cdot (f(r)\vec{r}) = 0$ show that $f(r) = \frac{c}{r^3}$ where c is an arbitrary constant.

Solution : $\nabla \cdot (f(r)\vec{r}) = \nabla(f(r)) \cdot \vec{r} + f(r)(\nabla \cdot \vec{r})$

$$\nabla(f(r)) = \frac{f'(r)}{r}\vec{r}\ \ (Verify)$$

$$\nabla\cdot(f(r)\vec{r}) = \frac{f'(r)}{r}\vec{r}\cdot\vec{r} + f(r)3 = \mathrm{r}f'(\mathrm{r}) + \mathrm{f}(\mathrm{r})3$$

Also, given $\nabla\cdot(f(r)\vec{r}) = 0$

Thus, $\mathrm{r}f'(\mathrm{r}) + \mathrm{f}(\mathrm{r})3 = 0$

$$\mathrm{r}f'(\mathrm{r}) = -\mathrm{f}(\mathrm{r})3$$

$$\frac{f'(r)}{f(r)} = -\frac{3}{r}$$

Integrating both sides with respect to r

$$\int\frac{f'(r)}{f(r)}dr = -\int\frac{3}{r}dr$$

$$\log f(r) = -3logr + logc = -logr^3 + logc = logc - logr^3 = \log c/r^3$$

Thus, $\log f(r) = log\frac{c}{r^3}$

Thus, $f(r) = \frac{c}{r^3}$ where c is an arbitrary constant.

Problem 24 : If ‘a’ is a constant vector and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then show that $\nabla\cdot\{(\vec{a}\cdot\vec{r})\vec{r}\} = 4(\vec{a}\cdot\vec{r})$.

Solution : Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\vec{a}\cdot\vec{r} = a_1x + a_2y + a_3z$$

Now, $\nabla\cdot\{(\vec{a}\cdot\vec{r})\vec{r}\} = \nabla\cdot\{(a_1x + a_2y + a_3z)\vec{r}\}$

$= \nabla(a_1x + a_2y + a_3z)\cdot\vec{r} + (a_1x + a_2y + a_3z)(\nabla\cdot\vec{\mathrm{r}})$ .... (1)

$$\nabla(a_1x + a_2y + a_3z) = \vec{i}\frac{\partial(a_1x + a_2y + a_3z)}{\partial x} + \vec{j}\frac{\partial(a_1x + a_2y + a_3z)}{\partial y} +$$

$$\vec{k}\frac{\partial(a_1x + a_2y + a_3z)}{\partial z}$$

$= a_1\vec{i} + a_2\vec{j} + a_3\vec{k} = \vec{a}$

Also, $\nabla\cdot\vec{\mathrm{r}} = 3$

$$\therefore (1) \Rightarrow \nabla\cdot\{(\vec{a}\cdot\vec{r})\vec{r}\} = \vec{a}\cdot\vec{r} + 3(\vec{a}\cdot\vec{r}) = 4(\vec{a}\cdot\vec{r})$$

Problem 25 : Find the value of 'a' if $\vec{A} = (axy - z^2)\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 - axz)\vec{k}$ is irrotational.

Solution : Given, $\vec{A}$ is irrotational.

$$\therefore \nabla \times \vec{A} = \vec{0}$$

$\therefore \nabla \times \vec{A} = (2y - 2y)\vec{i} - (-az + 2z)\vec{j} + (2x - ax)\vec{k}$ (Verify)

$$\nabla \times \vec{A} = \vec{0} \Rightarrow (2y - 2y)\vec{i} - (-az + 2z)\vec{j} + (2x - ax)\vec{k} = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$\therefore 2x - ax = 0$$

$$\therefore a = 2.$$

Problem 26 : Show that the following vector point functions are irrotational.

(i) $(4xy - z^3)\vec{i} + 2x^2\vec{j} - 3xz^2\vec{k}$

(ii) $(3x^2 + 2y^2 + 1)\vec{i} + (4xy - 3y^2z - 3)\vec{j} + (2 - y^3)\vec{k}$

(iii) $(y^2 + 2xz^2 - 1)\vec{i} + 2xy\vec{j} + 2x^2z\vec{k}$

Problem 27 : Show that the following vector $(y^2 - z^2 + 3yz - 2x)\vec{i} + (3xz + 2xy)\vec{j} + (3xy - 2xz + 2z)\vec{k}$ is both solenoidal and irrotational.

Problem 28 : If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ for all f(r), show that $\nabla \times \{f(r)\vec{r}\} = \vec{0}$.

Solution : We have, $\nabla \times (f(r)\vec{r}) = \nabla(f(r)) \times \vec{r} + f(r)(\nabla \times \vec{r})$

$$\nabla(f(r)) = \frac{f'(r)}{r}\vec{r} \quad (Verify)$$

$$\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{0} \text{ (verify)}$$

Thus, $\nabla \times (f(r)\vec{r}) = \frac{f'(r)}{r}\vec{r} \times \vec{r} + f(r)(\vec{0}) = \vec{0} + \vec{0} = \vec{0}$

Problem 29 : : If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ show that $\nabla \times (r^n\vec{r}) = \vec{0}$.

Solution : $\nabla \times (r^n\vec{r}) = \nabla(r^n) \times \vec{r} + r^n(\nabla \times \vec{r})$ .... (1)

$$\nabla(r^n) = \vec{i}\frac{\partial(r^n)}{\partial x} + \vec{j}\frac{\partial(r^n)}{\partial y} + \vec{k}\frac{\partial(r^n)}{\partial z}$$

$$= \vec{i}nr^{n-1}\frac{\partial r}{\partial x} + \vec{j}nr^{n-1}\frac{\partial r}{\partial y} + \vec{k}nr^{n-1}\frac{\partial r}{\partial z}$$

$$= \vec{i}nr^{n-1}\frac{x}{r} + \vec{j}nr^{n-1}\frac{y}{r} + \vec{k}nr^{n-1}\frac{z}{r}$$

$$= nr^{n-2}(\vec{i}x + \vec{j}y + \vec{k}z) = nr^{n-2}\vec{r}$$

$\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{0}$ (verify)

$\therefore (1) \Rightarrow \nabla \times (r^n\vec{r}) = \nabla(r^n) \times \vec{r} + r^n(\nabla \times \vec{r}) = nr^{n-2}\vec{r} \times \vec{r} + r^n\,(\vec{0}) \;\; = \vec{0}$

Problem 30 : Show that $\nabla \times \hat{r} = \vec{0}$

(Hint : Put $\hat{r} = \frac{1}{r}\vec{r}$)

Problem 31 : If $\vec{v} = \vec{w} \times \vec{r}$ where w is a constant vector and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$. Show that $\frac{1}{2} curl\vec{v} = \vec{w}$.

Solution : Let $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$

$$curl\vec{v} = \sum \vec{i} \times \frac{\partial \vec{v}}{\partial x} = \sum \vec{i} \times \frac{\partial(\vec{w} \times \vec{r})}{\partial x} = \sum \vec{i} \times \left(\vec{w} \times \frac{\partial(\vec{r})}{\partial x}\right)$$

$= \sum \vec{i} \times (\vec{w} \times \vec{i})$ [Since, $\frac{\partial \vec{r}}{\partial x} = \vec{i}$]

$$= \sum [(\vec{i} \cdot \vec{i})\vec{w} - (\vec{i} \cdot \vec{w})\vec{i}] = \sum (\vec{i} \cdot \vec{i})\vec{w} - \sum (\vec{i} \cdot \vec{w})\vec{i}$$

$= [(\vec{i} \cdot \vec{i})\vec{w} + (\vec{j} \cdot \vec{j})\vec{w} + (\vec{k} \cdot \vec{k})\vec{w}] - [(\vec{i} \cdot \vec{w})\vec{i} + (\vec{j} \cdot \vec{w})\vec{j} + (\vec{k} \cdot \vec{w})\vec{k}]$

$= (\vec{w} + \vec{w} + \vec{w}) - (w_1\vec{i} + w_2\vec{j} + w_3\vec{k}) = 3\vec{w} - \vec{w} = 2\vec{w}$

Thus, $curl\vec{v} = 2\vec{w}$

Hence, $\frac{1}{2} curl\vec{v} = \vec{w}$.

Problem 32 : If ‘a’ is a constant vector, show that

(i) $\nabla \times \{(\vec{a} \cdot \vec{r})\vec{r}\} = \vec{a} \times \vec{r}$

(ii) $\nabla \cdot \{\vec{a} \times \vec{r}\} = 0$

Solution :

(i) Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$$\nabla \times \{(\vec{a} \cdot \vec{r})\vec{r}\} = \{\nabla(\vec{a} \cdot \vec{r})\} \times \vec{r} + (\vec{a} \cdot \vec{r})(\nabla \times \vec{r})$$

$$\vec{a} \cdot \vec{r} = a_1x + a_2y + a_3z$$

$$\nabla(\vec{a}\cdot\vec{r}) = \vec{i}\frac{\partial(a_1x + a_2y + a_3z)}{\partial x} + \vec{j}\frac{\partial(a_1x + a_2y + a_3z)}{\partial y} +$$

$$\vec{k}\frac{\partial(a_1x + a_2y + a_3z)}{\partial z}$$

$= a_1\vec{i} + a_2\vec{j} + a_3\vec{k}=\vec{a}$

$$\nabla\times\vec{r} = \vec{0}$$

Thus,

$$\nabla\times\{(\vec{a}\cdot\vec{r})\vec{r}\} = \vec{a}\times\vec{r} + (\vec{a}\cdot\vec{r})\vec{0} = \vec{a}\times\vec{r}$$

(ii) $\vec{a}\times\vec{r} = 0$

Problem 33 : If $\vec{A}\ and\ \vec{B}$ are irrotational, show that $\vec{A}\times\vec{B}$ is solenoidal.

Solution : Given, $\vec{A}\ and\ \vec{B}$ are irrotational

Therefore,

$$\nabla\times\vec{A} = \vec{0}\ \&\ \nabla\times\vec{B} = \vec{0}$$

To prove, $\vec{A}\times\vec{B}$ is solenoidal

i.e., to prove $\nabla\cdot(\vec{A}\times\vec{B})$=0

Now, $\nabla\cdot\left(\vec{A}\times\vec{B}\right) = \left(\nabla\times\vec{A}\right)\cdot\vec{B} - \left(\nabla\times\vec{B}\right)\cdot\vec{A} = 0 - 0 = 0$

Hence, $\vec{A}\times\vec{B}$ is solenoidal.

Problem 34 : Show that $\nabla^2(logr) = \frac{1}{r^2}$.

Solution : $\nabla^2(logr) = \frac{\partial^2}{\partial x^2}(logr) + \frac{\partial^2}{\partial y^2}(logr) + \frac{\partial^2}{\partial z^2}(logr)$ .... (1)

Now, $\frac{\partial^2}{\partial x^2}(logr) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(logr)\right] = \frac{\partial}{\partial x}\left[\frac{1}{r}\frac{\partial r}{\partial x}\right] = \frac{\partial}{\partial x}\left[\frac{1}{r}\frac{x}{r}\right] = \frac{\partial}{\partial x}\left[\frac{x}{r^2}\right]$

$= -\frac{2x^2}{r^4} + \frac{1}{r^2}$ $\quad (verify)$

Similarly find $\frac{\partial^2}{\partial y^2}(logr)$ & $\frac{\partial^2}{\partial z^2}(logr)$.

Then Sub. all the values in (1),

Problem 35 : Show that $\nabla^2(r^n) = n(n+1)r^{n-2}$.

Solution : $\nabla^2(r^n) = \frac{\partial^2}{\partial x^2}(r^n) + \frac{\partial^2}{\partial y^2}(r^n) + \frac{\partial^2}{\partial z^2}(r^n)$ .... (1)

$$\frac{\partial^2}{\partial x^2}(r^n) = nx^2(n-2)r^{n-4} + nr^{n-2} \quad (verify)$$

$Similarly\ find\ \frac{\partial^2}{\partial y^2}(r^n)$ & $\frac{\partial^2}{\partial z^2}(r^n)$

$\therefore$ (1) => $\nabla^2(r^n) = nx^2(n-2)r^{n-4} + nr^{n-2} \quad + ny^2(n-2)r^{n-4} + nr^{n-2} \quad +$
$nz^2(n-2)r^{n-4} + nr^{n-2} \quad = n(n+1)r^{n-2}$.

Problem 36 : If $\emptyset$ is a harmonic function then show that $\nabla\emptyset$ is solenoidal.

Solution : Given, $\emptyset$ is a harmonic function.

$$\therefore We\ have, \nabla^2\emptyset = 0$$

To prove, $\nabla\emptyset$ is solenoidal

i.e., to prove, $\nabla \cdot \nabla\emptyset = 0$

$$\nabla \cdot \nabla\emptyset = \nabla^2\emptyset = 0$$

Hence, $\nabla\emptyset$ is solenoidal.

Problem 37: Show that $\nabla \cdot (\emptyset\nabla\psi - \psi\nabla\emptyset) = \emptyset\nabla^2\psi - \psi\nabla^2\emptyset$.

Solution : LHS=$\nabla \cdot (\emptyset\nabla\psi - \psi\nabla\emptyset) = \nabla \cdot (\emptyset\nabla\psi) - \nabla \cdot (\psi\nabla\emptyset)$ .... (1)

Consider, $\nabla \cdot (\emptyset\nabla\psi)$

Let $\vec{A} = \nabla\psi$

Now, $\nabla \cdot (\emptyset\vec{A}) = \nabla\emptyset \cdot \vec{A} + \emptyset(\nabla \cdot \vec{A}) = \nabla\emptyset \cdot \nabla\psi + \emptyset(\nabla \cdot \nabla\psi) = \nabla\emptyset \cdot \nabla\psi + \emptyset(\nabla^2\psi)$

Now, consider $\nabla \cdot (\psi\nabla\emptyset)$

Let $\vec{B} = \nabla\emptyset$

$$\nabla \cdot (\psi\vec{B}) = \nabla\psi \cdot \vec{B} + \psi(\nabla \cdot \vec{B}) = \nabla\psi \cdot \nabla\emptyset + \psi(\nabla \cdot \nabla\emptyset) = \nabla\psi \cdot \nabla\emptyset + \psi(\nabla^2\emptyset)$$

(1)=>, $\nabla \cdot (\emptyset\nabla\psi - \psi\nabla\emptyset) = \nabla\emptyset \cdot \nabla\psi + \emptyset(\nabla^2\psi) - \nabla\psi \cdot \nabla\emptyset - \psi(\nabla^2\emptyset)$

Thus, $\nabla \cdot (\emptyset\nabla\psi - \psi\nabla\emptyset) = \emptyset\nabla^2\psi - \psi\nabla^2\emptyset$.

Problem 38 : Show that $(\vec{V} \times \nabla) \times \vec{r} = -2\vec{V}$

Solution : Let $\vec{V} = V_1\vec{\imath} + V_2\vec{\jmath} + V_3\vec{k}$ & $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

First find $\vec{V} \times \nabla$ and then find $(\vec{V} \times \nabla) \times \vec{r}$

Problem 39 : Show that $(\vec{V} \cdot \nabla)\vec{V} = \frac{1}{2}\nabla V^2 - \vec{V} \times (\nabla \times \vec{V})$.

UNIT – II

EVALUATION OF DOUBLE AND TRIPLE INTEGRALS

Integration may be consider either as the inverse of differentiation or as the process of summation.

In calculus of a single variable the definite integral

$$\int_a^b f(x)dx$$

for $f(x) \geq 0$ is the area under the curve f(x) from x=a to x=b. For general f(x) the definite integral is equal to the area above the x-axis minus the area below the x-axis.

The multiple integral is a definite integral of a function of more than one real variable, for instance f(x,y) or f(x,y,z). Integrals of a function of two variables over a region in R^2 are called double integrals and integrals of a function of three variables over a region in R^3 are called triple integrals.

The definite integral can be extended to functions of more than one variable. Consider a function of 2 variables z=f(x,y). The definite integral is denoted by

$$\iint_R f(x,y)dA$$

where R is the region of integration in the xy-plane.

2.1 Evaluation of the Double integral

Given a double integral we can integrate the integrand f(x,y) with respect to x treating y as a constant and then integrating the resulting function with respect to y or vise versa.

Problems:

Problem 1: Evaluate $\int_0^1 \int_0^2 xydxdy$

Solution : Let $I = \int_{y=0}^{1} \int_{x=0}^{2} xydxdy = \int_{y=0}^{1} \left[\frac{x^2}{2}\right]_0^2 ydy = \int_{y=0}^{1} \left[\frac{4}{2} - 0\right] ydy$

$$= 2\int_{y=0}^{1} ydy = 2\left[\frac{y^2}{2}\right]_0^1 = 2\left[\frac{1}{2} - 0\right] = 1$$

Problem 2 : Evaluate $\int_0^a \int_0^b (x^2+y^2)dxdy$

Solution : Let $I = \int_{y=0}^{a} \int_{x=0}^{b} (x^2+y^2)dxdy$

$$= \int_{y=0}^{a} \left[\frac{x^3}{3} + y^2 x\right]_0^b dy$$

$$= \int_{y=0}^{a} \left[\frac{b^3}{3} + y^2 b - 0\right] dy$$

$$= \left[\frac{b^3}{3} y + b\frac{y^3}{3}\right]_0^a$$

$$= \left[\frac{b^3}{3} a + b\frac{a^3}{3} - 0\right]$$

$$= \frac{ab}{3}(a^2+b^2)$$

Problem 3 : Evaluate $\int_0^1 \int_0^2 (x+y)dxdy$ [Ans. : 3]

Problem 4 : Evaluate $\int_0^1 \int_0^{1-y} xdxdy$

Solution :

Let I=$\int_{y=0}^{1} \int_{x=0}^{1-y} xdxdy = \int_{y=0}^{1} \left[\frac{x^2}{2}\right]_0^{1-y} dy \quad = \int_{y=0}^{1} \left[\frac{(1-y)^2}{2} - 0\right] dy$

=1/2 $\int_{y=0}^{1} [1+y^2-2y]dy = \frac{1}{2}\left[y + \frac{y^3}{3} - 2\frac{y^2}{2}\right]_0^1 = \frac{1}{2}\left[1+\frac{1}{3}-1-0\right] = \frac{1}{6}$

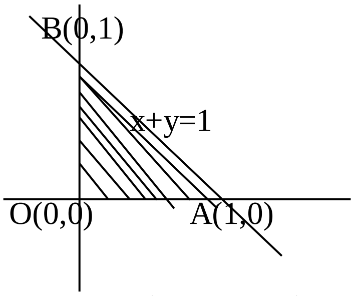
Problem 5 : Evaluate $\int_0^{4a} \int_{\frac{x^2}{4a}}^{x} x^2 ydxdy \quad \left[Ans.: \frac{1024a^5}{35}\right]$

Problem 6 : Evaluate $\int_0^a \int_{x^2}^{2x} (2x+3y)dydx$ $[Ans.: \frac{136}{15}]$

Problem 7 : Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^3 dydx \quad [Ans.: \frac{2}{15}a^5]$

Problem 8 : Evaluate $\iint xydxdy \; over$ the region bounded by the lines x=0; y=0; x+y=1.

Solution : x=0 is the y-axis, y=0 is the x-axis and x+y=1 is the line making intercepts with the x and y axis.



We have $x + y = 1 => y = 1 - x$

Keeping x as constant and y varies from 0 to 1-x.

x varies from 0 to 1.

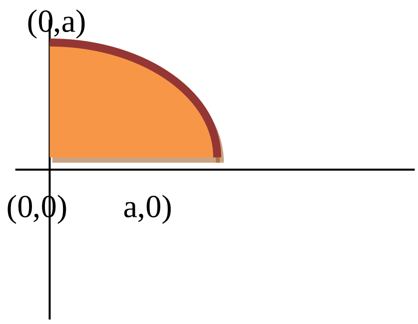
Thus, $\iint xydxdy = \int_{x=0}^{1}\int_{y=0}^{1-x} xydydx = \int_0^1 x\left[\frac{y^2}{2}\right]_0^{1-x} dx = \int_0^1 x[\frac{(1-x)^2}{2} - 0]dx$

$$= \frac{1}{2}\int_0^1 x\,(1 + x^2 - 2x)dx = \frac{1}{2}\int_0^1 (x + x^3 - 2x^2)dx$$

$=\frac{1}{2}\left[\frac{x^2}{2} + \frac{x^4}{4} - 2\,\frac{x^3}{3}\right]_0^1 = \frac{1}{2}\left[\frac{1}{2} + \frac{1}{4} - \frac{2}{3} - 0\right] = \frac{1}{2}\left[\frac{6+3-8}{12}\right] = \frac{1}{2} \times \frac{1}{12} = \frac{1}{24}.$

Problem 9 : Evaluate $\iint xydxdy$ taken over the <u>positive quadrant</u> of the circle $x^2 + y^2 = a^2$.

Solution :



Here x varies from 0 to a

y varies from 0 to $\sqrt{a^2 - x^2}$.

$\iint xydxdy = \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xydydx$

$$= \int_0^a x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a x\left[\frac{(\sqrt{a^2-x^2})^2}{2} - 0\right]dx$$

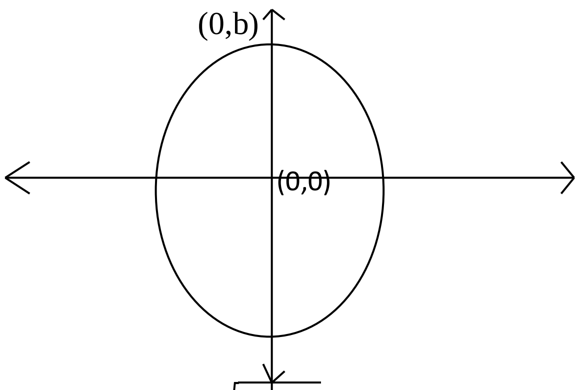
$$= \int_0^a x\left[\frac{a^2-x^2}{2}\right]dx = \frac{1}{2}\int_0^a (a^2x - x^3)dx$$

$$= \frac{1}{2}\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4} - 0\right] = \frac{1}{2}\left[\frac{2a^4 - a^4}{4}\right] = \frac{a^4}{8}.$$

Problem 10 : Evaluate $\iint (x^2+y^2)dxdy\ over$ the region for which x,y are each greater than or equal to 0 and $x + y \le 1$. (Ans. 1/6)

Problem 11 : Find the value of $\iint xy\, dxdy$ taken over the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution :



Solution : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 => y = b\sqrt{1 - \frac{x^2}{a^2}}$

x varies from 0 to a & y varies from 0 to $b\sqrt{1 - \frac{x^2}{a^2}}$

Thus, $\iint xy\, dxdy = \int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} xydydx$=$\frac{a^2b^2}{8}$(verify).

2.2 Changing the order of integration

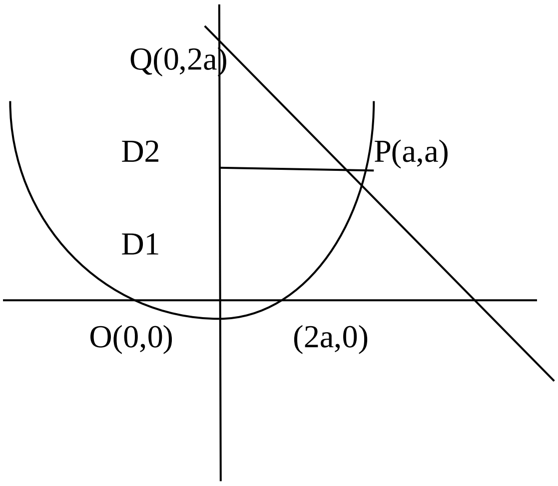
For a given integral in a region, change of order changes the limits of x and y.

Problem 12 : Change the order of integration in the integral $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xydxdy$ and evaluate it.

Solution : In the given integral y varies from $\frac{x^2}{a}$ to 2a-x and x varies from 0 to a.

So, we have y=$\frac{x^2}{a}$ $=> x^2 = ay$ &

y=2a-x=> x+y=2a



Here the region of integration is OPQ.

In changing the order of integration, we integrate first with respect to x keeping y as constant.

Here the region is divided into 2 parts D1 and D2.

Therefore, $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xydxdy \ = \iint_{D1} xydxdy + \iint_{D2} xydxdy$................ (1)

Consider the region D1.

y varies from 0 to a and x varies from 0 to $\sqrt{ay}$.

$$\iint_{D1} xydxdy = \int_{y=0}^{a} \int_{x=0}^{\sqrt{ay}} xy\, dxdy = \frac{a^4}{6}.(verify)$$

Consider D2.

y varies from a to 2a and x varies from 0 to 2a-y

$$\iint_{D2} xydxdy = \int_{y=a}^{2a} \int_{x=0}^{2a-y} xy\, dxdy = \frac{5}{24}a^4.$$

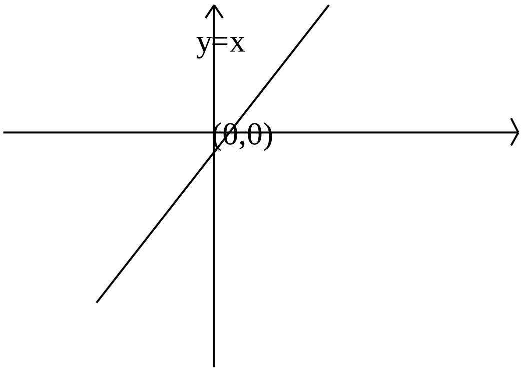
Thus (1)=> $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xydxdy \ = \frac{a^4}{6} + \frac{5}{24}a^4 = \frac{3}{8}a^4.$

Problem 13 : By changing the order of integration evaluate $\int_{x=0}^{\infty}\int_{y=x}^{\infty}\frac{e^{-y}}{y}dxdy$.

Solution :

Here x varies from 0 to ∞ and y varies from x to ∞.

The regions are x=0; x=∞ ; y=x; y=∞.



After changing the order of integration y varies for 0 to ∞ and x varies from 0 to y.

$$\int_{x=0}^{\infty}\int_{y=x}^{\infty}\frac{e^{-y}}{y}dxdy = \int_{y=0}^{\infty}\int_{x=0}^{y}\frac{e^{-y}}{y}dxdy$$

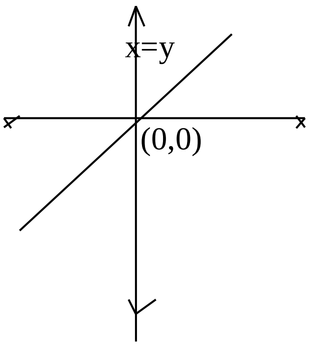
$$= \int_{y=0}^{\infty}\frac{e^{-y}}{y}[x]_0^y dy$$

$= \left[\frac{e^{-y}}{-1}\right]_0^{\infty} = -[e^{-\infty} - e^{0}] = 1$

Problem 14 : Changing the order of integration evaluate $\int_0^a\int_y^a\frac{x}{x^2+y^2}dxdy$.

Solution : Here x varies from y to a and y varies from 0 to a.

The regions are x=y, x=a, y=0, y=a.



By changing the order of integration, x varies from 0 to a & y varies from 0 to x.

$$\therefore \int_0^a \int_y^a \frac{x}{x^2+y^2}\,dxdy = \int_{x=0}^{a} \int_{y=0}^{x} \frac{x}{x^2+y^2}\,dydx$$

$$= \int_{x=0}^{a} x\left[\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right]_0^x dx$$

$$= \int_{x=0}^{a} \frac{x1}{x}\left(\tan^{-1}\left(\frac{x}{x}\right) - \tan^{-1}(0)\right)dx$$

$$= \int_{x=0}^{a} \left(\tan^{-1}(1) - \tan^{-1}(0)\right)dx$$

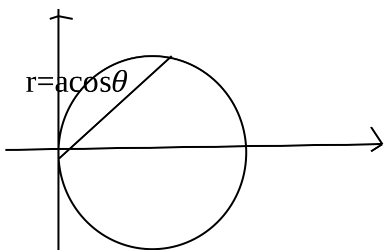
$$= \int_{x=0}^{a} \frac{\pi}{4}dx = \frac{\pi}{4}[x]_0^a = \frac{\pi a}{4}.$$

2.3 Double integral in polar co-ordinates

The double integral in Cartesian form $\iint_R f(x,y)dxdy\ transforms$ into $\iint_R f(rcos\theta, rsin\theta)rdrd\theta.$

Problem 15: Evaluate $\iint r\sqrt{a^2-r^2}drd\theta$ over the upper half of the circle $r = acos\theta.$

Solution :



Here r varies from 0 to $acos\theta$ and θ varies from 0 $to\ \pi/2.$

$$\therefore \iint_R r\sqrt{a^2-r^2}drd\theta = \int_{\theta=0}^{\pi/2} \int_{r=0}^{acos\theta} r\sqrt{a^2-r^2}drd\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{acos\theta} \sqrt{a^2-r^2}\left(-\frac{d(a^2-r^2)}{2}\right)d\theta \qquad [Since, d(a^2-r^2) = -2rdr =>$$

$$\therefore\ rdr = -\frac{d(a^2-r^2)}{2}\ ]$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{acos\theta} (a^2 - r^2)^{1/2} \left(-\frac{d(a^2-r^2)}{2}\right) d\theta$$

$$= -\frac{1}{2} \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{acos\theta} (a^2 - r^2)^{1/2} d(a^2 - r^2) d\theta$$

$$= -\frac{1}{2} \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{(a^2 - r^2)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^{acos\theta} d\theta$$

$$= -\frac{1}{2} \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{(a^2 - r^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{acos\theta} d\theta$$

$$= -\frac{1}{2} \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{[a^2 - (acos\theta)^2]^{\frac{3}{2}} - [a^2 - 0]^{\frac{3}{2}}}{\frac{3}{2}}\right] d\theta$$

$$= -\frac{1}{2} \times \frac{2}{3} \int_{\theta=0}^{\frac{\pi}{2}} ([a^2 - a^2cos^2\theta]^{\frac{3}{2}} - [a^2]^{\frac{3}{2}}) d\theta$$

$$= -\frac{1}{3} \int_{\theta=0}^{\frac{\pi}{2}} \{[a^2(1 - cos^2\theta)]^{\frac{3}{2}} - [a^2]^{\frac{3}{2}}\} \, d\theta$$

$$= -\frac{1}{3} \int_{\theta=0}^{\frac{\pi}{2}} \{[a^2]^{\frac{3}{2}} (sin^2\theta)^{\frac{3}{2}} - a^3\} \, d\theta$$

$$= -\frac{1}{3} \int_{\theta=0}^{\frac{\pi}{2}} \{a^3 sin^3\theta - a^3\} \, d\theta$$

$$= -\frac{1}{3} \left[\int_{\theta=0}^{\frac{\pi}{2}} a^3 sin^3\theta \, d\theta - \int_{\theta=0}^{\frac{\pi}{2}} a^3 \, d\theta] \right]$$

$$= -\frac{1}{3} \left[a^3 \left[\frac{2}{3}\right] - a^3 [\theta]_0^{\pi/2}\right]$$

$$= \frac{3\pi - 4}{18} a^3 \ (verify)$$

Problem 16 : By transforming into polar co-ordinates evaluate $\iint \frac{x^2y^2}{x^2+y^2} dxdy$ over the annular region between the circle $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ (b>a).

Solution : Put $x = rcos\theta; y = rsin\theta\ and\ dxdy = rdrd\theta$

r varies from a to b and θ varies from $0\ to\ 2\pi$.

$$\iint \frac{x^2y^2}{x^2+y^2} dxdy = \int_{\theta=0}^{2\pi} \int_{r=a}^{b} \frac{r^2cos^2\theta r^2sin^2\theta}{r^2cos^2\theta + r^2sin^2\theta} rdrd\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=a}^{b} \frac{r^5cos^2\theta sin^2\theta}{r^2(cos^2\theta + sin^2\theta)} drd\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=a}^{b} r^3cos^2\theta sin^2\theta drd\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\frac{r^4}{4}\right]_a^b cos^2\theta sin^2\theta d\theta$$

$$= \frac{1}{4} \int_{\theta=0}^{2\pi} [b^4 - a^4] cos^2\theta sin^2\theta d\theta$$

$$= \frac{[b^4 - a^4]}{4} \int_{\theta=0}^{2\pi} cos^2\theta(1 - cos^2\theta) d\theta$$

$$= \frac{[b^4 - a^4]}{4} \int_{\theta=0}^{2\pi} (cos^2\theta - cos^4\theta) d\theta$$

$$= \frac{[b^4 - a^4]}{4} \left[\int_{\theta=0}^{2\pi} cos^2\theta d\theta - \int_{\theta=0}^{2\pi} cos^4\theta d\theta\right]$$

$$= \frac{[b^4 - a^4]}{4} \left[4\int_{\theta=0}^{\pi/2} cos^2\theta d\theta - 4\int_{\theta=0}^{\pi/2} cos^4\theta d\theta\right]$$

$$= \frac{4[b^4 - a^4]}{4} \left[\int_{\theta=0}^{\pi/2} cos^2\theta d\theta - \int_{\theta=0}^{\pi/2} cos^4\theta d\theta\right]$$

$$= [b^4 - a^4]\left[\frac{1}{2}\frac{\pi}{2} - \frac{3}{4}\frac{1}{2}\frac{\pi}{2}\right]$$

$$= [b^4 - a^4]\left[\frac{4\pi - 3\pi}{16}\right]$$

$$= [b^4 - a^4]\left[\frac{\pi}{16}\right]$$

Problem 17 : By changing into polar co-ordinates evaluate the integral $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}(x^2 + y^2)dxdy$.

Solution : Here x=0, x=2a, y=0, y= $\sqrt{2ax - x^2} \Rightarrow y^2 = 2ax - x^2$

$$i.e., x^2 + y^2 = 2ax$$

Put $x = rcos\theta; y = rsin\theta\ and\ dxdy = rdrd\theta$

$$r^2cos^2\theta + r^2sin^2\theta = 2arcos\theta$$

$$i.e., r^2 = 2arcos\theta \Rightarrow r = 2acos\theta$$

r varies from 0 to $2acos\theta$ & θ *varies from* 0 *to* $\pi/2$.

$$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}(x^2 + y^2)dxdy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{2acos\theta}(r^2cos^2\theta + r^2sin^2\theta)\ rdrd\theta$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{2acos\theta} r^2\ rdrd\theta = \int_{\theta=0}^{\pi/2}\int_{r=0}^{2acos\theta} r^3\ drd\theta$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{2acos\theta} r^3\ drd\theta = \int_{\theta=0}^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2acos\theta} d\theta$$

$$= \int_{\theta=0}^{\pi/2}\left[\frac{(2acos\theta)^4 - 0}{4}\right]d\theta$$

$$= \int_{\theta=0}^{\pi/2}\left[\frac{16a^4cos^4\theta}{4}\right]d\theta = 4a^4\int_0^{\pi/2} cos^4\theta\ d\theta$$

$$= 4a^4\frac{3}{4}\frac{1}{2}\frac{\pi}{2} = \frac{3\pi a^4}{4}$$

2.4 Triple Integrals

The triple integral is defined in a manner similar to that of the double integral, if f(x,y,z) is continuous and single valued function of x, y and z over the region of space R enclosed by the surface, then

$\iiint_V f(x,y,z)dV = \iiint f(x,y,z)dxdydz.$

Notes : (1) $\iint_R dxdy\ represents$ the area of the region R.

(2) $\iiint_D dxdydz$ represents the volume of the region D.

Problem 18 : Evaluate $\int_0^2 \int_1^3 \int_1^2 xy^2\ zdzdydx$

Solution : $\int_{x=0}^2 \int_{y=1}^3 \int_{z=1}^2 xy^2\ zdzdydx = \int_{x=0}^2 \int_{y=1}^3 xy^2 \left[\frac{z^2}{2}\right]_1^2 dydx$

$$= \int_{x=0}^{2} \int_{y=1}^{3} xy^2 \left[\frac{4}{2} - \frac{1}{2}\right] dydx$$

$$= \frac{3}{2} \int_{x=0}^{2} x \left[\frac{y^3}{3}\right]_1^3 dx$$

$$= \frac{3}{2} \int_{x=0}^{2} x \left[\frac{27}{3} - \frac{1}{3}\right] dx$$

$$= \frac{3}{2} \times \frac{26}{3} \int_{x=0}^{2} x\, dx$$

$$= 13 \left[\frac{x^2}{2}\right]_0^2 = 13 \times \frac{4}{2} = 26.$$

Problem 19: Evaluate $I = \int_0^{loga} \int_0^x \int_0^{x+y} e^{x+y+z} dxdydz.$

Solution : $I = \int_{x=0}^{loga} \int_{y=0}^{x} \int_{z=0}^{x+y} e^{x+y+z} dzdydx = \int_0^{loga} \int_0^x [e^{x+y+z}]_0^{x+y} dydx$

$$= \int_{0}^{loga} \int_{0}^{x} [e^{x+y+x+y} - e^{x+y+0}] dydx$$

$$= \int_0^{loga} \int_0^{x} [e^{2(x+y)} - e^{x+y}] dydx$$

$$= \int_0^{loga} \left[\frac{e^{2(x+y)}}{2} - e^{(x+y)}\right]_0^x dx$$

$$= \int_0^{loga} \left\{\left[\frac{e^{2(x+x)}}{2} - e^{(x+x)}\right] - \left[\frac{e^{2(x+0)}}{2} - e^{(x+0)}\right]\right\} dx$$

$$= \int_0^{loga} \left\{\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^x\right\} dx$$

$$= \int_0^{loga} \left\{\frac{e^{4x}}{2} - 3\frac{e^{2x}}{2} + e^x\right\} dx$$

$$= \left[\frac{e^{4x}}{8} - 3\frac{e^{2x}}{4} + e^x\right]_0^{loga}$$

$$= \left[\frac{e^{4loga}}{8} - 3\frac{e^{2loga}}{4} + e^{loga}\right] - \left[\frac{e^0}{8} - 3\frac{e^0}{4} + e^0\right]$$

$$= \left[\frac{e^{loga^4}}{8} - 3\frac{e^{loga^2}}{4} + e^{loga}\right] - \left[\frac{1}{8} - 3\frac{1}{4} + 1\right]$$

$$= \frac{a^4}{8} - \frac{3a^2}{4} + a - \left[\frac{1-6+8}{8}\right]$$

$$= \frac{a^4 - 6a^2 + 8a}{8} - \frac{3}{8}$$

Problem 20 : Evaluate $\iiint xyzdxdydz$ taken through the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution : Put y=z=0 ∴ $x^2 = a^2 => x = \pm a$

Put z=0 ∴ $x^2 + y^2 = a^2 => y^2 = a^2 - x^2 => y = \pm\sqrt{a^2 - x^2}$

$$x^2 + y^2 + z^2 = a^2 => z = \pm\sqrt{a^2 - x^2 - y^2}$$

$$\iiint xyzdxdydz = \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} xyzdzdydx$$

$$= \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} \left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} xydydx$$

$$= \frac{1}{2}\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy(a^2-x^2-y^2)dydx$$

$$= \frac{a^6}{48}\ (verify)$$

Problem 21 : Evaluate $\iiint \frac{dxdydz}{(x+y+z+1)^3}$ taken over the volume bounded by the planes x=0; y=0; z=0; x+y+z=1.

Solution : Here x varies from 0 to 1

y varies from 0 to 1-x and z varies from 0 to 1-x-y.

$$\iiint \frac{dxdydz}{(x+y+z+1)^3} = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} (x+y+z+1)^{-3}dzdydx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} \left[\frac{(x+y+z+1)^{-3+1}}{-3+1}\right]_0^{1-x-y} dydx$$

$$= -\frac{1}{2}\int_{x=0}^{1}\int_{y=0}^{1-x} [(x+y+z+1)^{-2}]_0^{1-x-y}dydx$$

$$= -\frac{1}{2}\int_{x=0}^{1}\int_{y=0}^{1-x} [(x+y+1-x-y+1)^{-2}-(x+y+0+1)^{-2}]\,dydx$$

$$= -\frac{1}{2}\int_{x=0}^{1}\int_{y=0}^{1-x} [(2)^{-2}-(x+y+1)^{-2}]\,dydx$$

$$= -\frac{1}{2}\int_{x=0}^{1} \left[\frac{1}{4}y-\frac{(x+y+1)^{-2+1}}{-2+1}\right]_0^{1-x} dx$$

$$= -\frac{1}{2}\int_{x=0}^{1}\left[\frac{1}{4}y - \frac{(x+y+1)^{-1}}{-1}\right]_{0}^{1-x} dx$$

$$= -\frac{1}{2}\int_{x=0}^{1}\left[\frac{1}{4}y + (x+y+1)^{-1}\right]_{0}^{1-x} dx$$

$$= -\frac{1}{2}\int_{x=0}^{1}\left\{\left[\frac{1}{4}(1-x) + (x+1-x+1)^{-1}\right] - [0 + (x+0+1)^{-1}]\right\} dx$$

$$= -\frac{1}{2}\int_{x=0}^{1}\left\{\left[\frac{1}{4}(1-x) + (2)^{-1}\right] - [(x+1)^{-1}]\right\} dx$$

$$= -\frac{1}{2}\int_{x=0}^{1}\left\{\left[\frac{1}{4}(1-x) + (2)^{-1}\right] - [(x+1)^{-1}]\right\} dx$$

$$= -\frac{1}{2}\int_{x=0}^{1}\left\{\frac{1}{4} - \frac{1}{4}x + \frac{1}{2} - \frac{1}{(x+1)}\right\} dx$$

$$= -\frac{1}{2}\int_{x=0}^{1}\left\{\frac{3}{4} - \frac{1}{4}x - \frac{1}{(x+1)}\right\} dx$$

$$= -\frac{1}{2}\left[\frac{3}{4}x - \frac{1}{4}\frac{x^2}{2} - \log(x+1)\right]_{0}^{1}$$

$$= -\frac{1}{2}\left\{\left[\frac{3}{4} - \frac{1}{4}\frac{1}{2} - \log(1+1)\right] - \left[\frac{3}{4}0 - \frac{1}{4}\frac{0}{2} - \log(0+1)\right]\right\}$$

$$= -\frac{1}{2}\left\{\left[\frac{3}{4} - \frac{1}{8} - \log 2\right] - [-\log(1)]\right\}$$

$$= -\frac{1}{2}\left\{\left[\frac{6-1}{8} - \log 2\right] + 0\right\}$$

$$= -\frac{1}{2}\left[\frac{5}{8} - log2\right] = \frac{1}{2}log2 - \frac{5}{16}$$

UNIT III

VECTOR INTEGRATION – LINE, SURFACE AND VOLUME INTEGRALS

3.1 LINE INTEGRALS

Another way of generalising the Riemann integral $\int_a^b f(x)dx\ is$ by placing the interval [a, b] by a curve in R^3. In this generalization the integrand is a vector valued function $f = f_1\vec{\imath} + f_2\vec{\jmath} + f_3\vec{k}$.

Definition:

Let C be a curve in R^3 described by a continuous vector valued function $r = r(t) = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$ where $a \leq t \leq b$.

Let $f = f_1(x,y,z)\vec{\imath} + f_2(x,y,z)\vec{\jmath} + f_3(x,y,z)\vec{k}$ be a continuous function defined in some region which contains the curve C. The **line integral of f over C** denoted by $\int_C f dr$ is defined by

$$\int_C f \cdot dr = \int_a^b [f_1[x(t),y(t),z(t)]x'(t) + f_2[x(t),y(t),z(t)]y'(t) + f_3[x(t),y(t),z(t)]z'(t)]dt.$$

Solved Problems

Problem 1: Evaluate $\int_C f \cdot dr$ where $f = (x^2 + y^2)\vec{\imath} + (x^2 - y^2)\vec{\jmath}$ and C is the curve $y = x^2$ joining (0,0) and (1,1).

Solution : The parametric equation of the curve can be taken as $x = t, y = t^2 , 0 \leq t \leq 1$.

$$\int_C f \cdot dr = \int_0^1 [(t^2 + t^4)1 + (t^2 - t^4)2t]dt = \int_0^1 [t^2 + t^4 + 2t^3 - 2t^5]dt$$

$$= \left[\frac{1}{3}t^3 + \frac{t^5}{5} + \frac{2t^4}{4} - \frac{2t^6}{6}\right]_0^1 = \frac{1}{3} + \frac{1}{5} + \frac{1}{2} - \frac{1}{3} = \frac{2+5}{10} = \frac{7}{10}.$$

Problem 2: If $f = x^2 i - xyj\ and\ C$ is the straight line joining the points (0,0) and (1,1) find $\int_C f \cdot dr$.

Solution : The equation of the given line is y=x and its parametric equation can be taken as $x = t, y = t\ where\ 0 \leq t \leq 1$.

$$\therefore \int_C f \cdot dr = \int_0^1 (t^2 - t^2)dt = 0.$$

Problem 3: Evaluate $\int_C f \cdot dr$where $f = (x^2 + y^2)i - 2xyj$ and the curve C is the rectangle in the x-y plane bounded by $y = 0, y = b, x = 0, x = a$.

Solution : Let O=(0,0), A=(a,0), B=(a,b) and C = (0,b) be the vertices of the given rectangle.

$$\therefore \int_C f \cdot dr = \int_{OA} f \cdot dr + \int_{AB} f \cdot dr + \int_{BC} f \cdot dr + \int_{CO} f \cdot dr$$

Now the parametric equation of OA can be taken as $x = t, y = 0 \; where \; 0 \le t \le a$.

$$\therefore \int_{OA} f \cdot dr = \int_0^a t^2 dt = \frac{a^3}{3}.$$

Now the parametric equation of AB can be taken as $x = a, y = t \; where \; 0 \le t \le b$.

$$\therefore \int_{AB} f \cdot dr = \int_0^b -2atdt = -ab^2.$$

Now the parametric equation of BC can be taken as $x = t, y = b \; where \; 0 \le t \le a$.

$$\therefore \int_{BC} f \cdot dr = \int_0^a (t^2 + b^2)dt = -\left(\frac{a^3}{3} + ab^2\right).$$

Now the parametric equation of CO can be taken as $x = 0, y = t \; where \; 0 \le t \le b$.

$$\therefore \int_{CO} f \cdot dr = -\int_0^b 0dt = 0.$$

$$\therefore \int_C f \cdot dr = \frac{a^3}{3} - ab^2 - \left(\frac{a^3}{3} + ab^2\right) + 0 = -2ab^2.$$

Problem 4: If $f = (2y + 3)i + xzj + (yz - x)k$ evaluate $\int_C f \cdot dr$ along the following paths C. (i) $x = 2t^2; y = t; z = t^3 \;\; from \; t = 0 \; to \; t = 1$.

(ii)The polygonal path P consisting of the three line segments AB, BC and CD where A=(0,0,0), B=(0,0,1), C=(0,1,1) and D=(2,1,1).

(iii) The straight line joining (0,0,0) and (2,1,1).

Solution : (i) $\int_C f \cdot dr = \int_0^1 [(2t + 3)4t + 2t^5 + (t^4 - 2t^2)3t^2] \, dt$

$$= \left[\frac{8}{3}t^3 + 6t^2 + \frac{1}{3}t^6 + \frac{3}{7}t^7 - \frac{6}{5}t^5\right]_0^1 = \frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} + \frac{6}{5} = \frac{288}{85}.$$

(ii) $\int_P f \cdot dr = \int_{AB} f \cdot dr + \int_{BC} f \cdot dr + \int_{CD} f \cdot dr$

$$\int_{AB} f \cdot dr = \int_0^1 0\, dt = 0\ [since, x = 0, y = 0, z = t\ and\ 0 \le t \le 1\ along\ AB]$$

$$\int_{BC} f \cdot dr = \int_0^1 0\, dt = 0\ [since, x = 0, y = t, z = 1\ and\ 0 \le t \le 1\ along\ BC]$$

$$\int_{CD} f \cdot dr = \int_0^2 5\, dt = 0\ [since, x = 1, y = 1, z = t\ and\ 0 \le t \le 2\ along\ CD]$$

$$= [5t]_0^3 = 10.$$

Thus $\int_P f \cdot dr = 10.$

(iii)The parametric equation of the line joining the points (0,0,0) and (2,1,1) can be taken as $x = 2t, y = t, z = t\ where\ 0 \le t \le 1.$

Thus, $\int_C f \cdot dr = \int_0^1 (2t+3)2 + t^2 + (t^2 - 2t)dt = \int_0^1 (3t^2 + 2t + 6)dt = [t^3 + t^2 + 6t]_0^1 = 8.$

Exercises

1. Evaluate $\int_{(1,1)}^{(4,2)} f \cdot dr$ where $f = (x+y)i + (y-x)j$ along (i) the parabola $y^2 = x$

(ii)the straight line joining (1,1) and (4,2).

2. If $f = (x^2 - y^2)i + 2xyj$, evaluate $\int_C f \cdot dr$ along the curve C in the x-y plane given by $y = x^2 - x$ from the point (1,0) to (2,2).
3. Evaluate $\int_C f \cdot dr$ where $f = (x-y)i + (y-2x)j$ and C is the closed curve in the x-y plane $x = 2cost, y = 3sint\ from\ t = 0\ to\ t = 2\pi.$

3.2 SURFACE INTEGRAL

Definition : Consider a surface S. Let n denote the unit outward normal to the surface S. Let R be the projection of the surface S on the x-y plane. Let f be a vector valued function defined in some region containing the surface S.

Then the surface integral of f over S is defined to be

$$\iint_S f \cdot n\, dS = \iint_R \frac{f \cdot n}{|n \cdot k|} dx\, xy.$$

Note : We can also define surface integral by considering the projection of the surface on the y-z plane or z-x plane.

Problem 1 : Evaluate $\iint_S f \cdot n\, dS$ where $f = (x + y^2)i - 2xj + 2yz\, k$ and S is the surface of the plance $2x + y + 2z = 6$ in the first octant.

Solution : Let $\emptyset(x, y, z) = 2x + y + 2z - 6$

The unit surface normal $n = \frac{\nabla\emptyset}{|\nabla\emptyset|} = \frac{2i+j+2k}{3}$.

$$\therefore f \cdot n = \frac{1}{3}[2(x + y^2) - 2x + 4yz]$$

$$= \frac{1}{3}[2(x + y^2) - 2x + 2y(6 - 2x - y)]$$

$$= \frac{4}{3}[3y - xy].$$

$$\therefore \frac{f \cdot n}{|n \cdot k|} = 2(3y - xy).$$

The projection of the surface on the x-y plane is the region R bounded by the axes and the straight line 2x+y=6 as shown in the figure.

$$\iint_S f \cdot n\, dS = \iint_R 2(3y - xy)dx\, dy$$

$$= 2\int_0^3 \int_0^{6-2x} (3y - xy)dy\, dx = 2\int_0^3 \left[\frac{3}{2}y^2 - \frac{1}{2}xy^2\right]_0^{6-2x} dx$$

$$= 2\int_0^3 \left[\frac{3}{2}(6 - 2x)^2 - \frac{1}{2}x(6 - 2x)^2\right] dx$$

$$= \left[-\frac{1}{2}(6 - 2x)^3 - 18x^2 - x^4 + 8x^3\right]_0^3$$

$$= 81.$$

Problem 2 : Evaluate $\iint_S (\nabla \times f) \cdot n\, dS$where $f = y^2i + yj - xzk$ and S is the upper half of the sphere $x^2 + y^2 + z^2 = a^2\ and\ z \geq 0$.

Solution : Let $\emptyset(x, y, z) = x^2 + y^2 + z^2 - a^2$.

The unit surface normal n is given by $n = \frac{\nabla\emptyset}{|\nabla\emptyset|} = \frac{2xi+2yj+2zk}{2\sqrt{x^2+y^2+z^2}} = \frac{1}{a}(xi + yj + zk)$

Also $\nabla \times f = zj - 2yk$.

$$(\nabla \times f) \cdot n = \left(\frac{1}{a}\right)(yz - 2yz) = -\left(\frac{1}{a}\right)yz.$$

Also, $n \cdot k = \left(\frac{1}{a}\right)z$.

$$\therefore \frac{\nabla \times f}{|n \cdot k|} = -y.$$

The projection of the surface on the x-y plane is the circle $x^2 + y^2 = a^2$. Let R denote the interior of the circle.

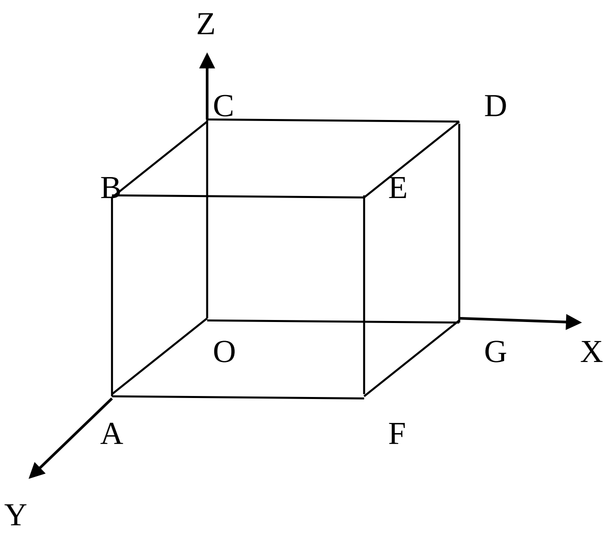
$$\iint_S (\nabla \times f) \cdot n \, dS = -\iint_y y \, dx \; dy$$

$$Put\ x = rcos\theta\ and\ y = rsin\theta.\ Hence, |J| = r.$$

$$\iint_S (\nabla \times f) \cdot n \, dS = -\int_0^{2\pi} \int_0^a rsin\theta r \, dr \, d\theta = -\int_0^{2\pi} \frac{1}{3} a^3 sin\theta d\theta = 0.$$

Problem 3 : Evaluate $\iint f \cdot ndS$ where $f = (x^3 - \mathrm{yz})\mathrm{i} - 2\mathrm{x}^2\mathrm{yj} + 2\mathrm{k}$ and S is the surface of the cube bounded by $x = 0, y = 0, z = 0, x = a, y = a\ and\ z = a$.

Solution:



On the face $OABC, n = -i\ and\ x = 0$.

$$\therefore \iint_{OABC} f \cdot n \, dS = \int_0^a \int_0^a yz \, dydz = \int_0^a z \left[\frac{y^2}{2}\right]_0^a dz = \int_0^a z \left[\frac{a^2}{2} - 0\right] dz$$

$$= \frac{a^2}{2}\left[\frac{z^2}{2}\right]_0^a = \frac{a^4}{4}.$$

On the face DEFG, $n = i \; and \; x = a$.

$$\therefore \iint_{DEFG} f \cdot n \, dS = \int_0^a \int_0^a (a^3 - yz) \, dydz = \int_0^a \left[a^4 - \frac{a^2 z}{2}\right] dz$$

$$= a^5 - \frac{1}{4}a^4.$$

On the face $OGDC, n = -j \; and \; y = 0$.

$$\therefore \iint_{OGDC} f \cdot n \, dS = \int_0^a \int_0^a 0 \, dxdz = 0.$$

On the face AFEB, $n = j \; and \; y = a$.

$$\therefore \iint_{AFEB} f \cdot n \, dS = \int_0^a \int_0^a -2x^2 a \, dxdz = \int_0^a -2x^2 a^2 dz$$

$$= -\frac{2}{3}a^5.$$

On the face $OAFG, n = -k \; and \; z = 0$.

$$\therefore \iint_{OAFG} f \cdot n \, dS = -\int_0^a \int_0^a 2 \, dxdy = -\, 2\, a^2.$$

On the face $CBED, n = k \; and \; z = a$.

$$\therefore \iint_{CBED} f \cdot n \, dS = \int_0^a \int_0^a 2dxdy = 2a^2.$$

$$\therefore \iint_S f \cdot ndS = \frac{a^4}{4} + a^5 - \frac{1}{4}a^4 + 0 - \frac{2}{3}a^5 - 2a^2 + 2a^2 = \frac{a^5}{3}.$$

Problem 4: Evaluate $\iint_S (x^2 + y^2) dS \; where$ S is the Surface of the cone $z^2 = 3(x^2 + y^2)$ bounded by $z = 0 \; and \; z = 3$.

UNIT IV

GREEN'S, STOKE'S AND DIVERGENCE THEOREM

4.1 Green's theorem in plane

If C is a simple closed curve in the xy plane bounding an area R and

M(x,y) and N(x,y) are continuous functions of x and y having continuous derivatives in R, then

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

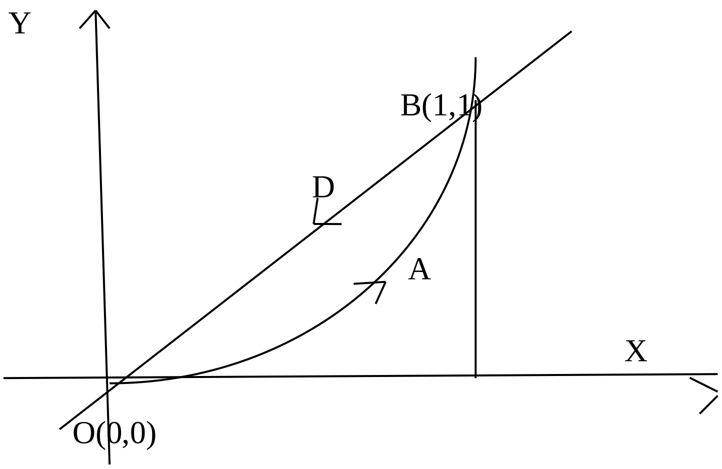
Problem 1 : Verify Green's theorem in plane for the integral $\int_C (xy + y^2)dx + x^2 dy$, where C is the curve enclosing the region R bounded by the parabola $y = x^2$ and the line $y = x$.

Solution : Given $y = x^2$ and $y = x$.

Therefore, $x = x^2 \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0 \Rightarrow x = 0 \; or \; x = 1$

When x=0, y=0 & when x=1, y=1

Thus the parabola and the line intersect at (0,0) and (1,1).



In the figure OABDO, the curve C consists of the parabolic arc OAB and the line segment BDO.

The parametric equations of OAB are $x = t, y = t^2$ where t varies from 0 to 1.

Here, $M = xy + y^2$ & $N = x^2$

$$\therefore M = t \times t^2 + t^4 = t^3 + t^4 \; \& \; N = t^2$$

$$dx = dt \; \& \; dy = 2tdt$$

$$\int_C (xy+y^2)dx + x^2dy = \int_0^1 (t^3+t^4)dt + t^2 2tdt = \int_0^1 (t^3+t^4+2t^3)dt$$

$= \left[\frac{t^4}{4}+\frac{t^5}{5}+\frac{2t^4}{4}\right]_0^1 = \frac{1}{4}+\frac{1}{5}+\frac{1}{2}-0 = \frac{19}{20}$

The parametric equations of BDO are $x = t, y = t$ where t varies from 1 to 0.

$$\therefore M = t^2+t^2 = 2t^2 \;\&\; N = t^2$$

$$dx = dt \;\&\; dy = dt$$

$$\int_C (xy+y^2)dx + x^2dy = \int_1^0 (2t^2)dt + t^2dt = \int_1^0 (2t^2+t^2)dt$$

$= \left[\frac{3t^3}{3}\right]_1^0 = -1.$

Hence,

$$\int_C (xy+y^2)dx + x^2dy = \int_{OAB} (xy+y^2)dx + x^2dy + \int_{BDO} (xy+y^2)dx + x^2dy$$

$= \frac{19}{20} - 1 = -\frac{1}{20}$ .... (1)

$$\frac{\partial N}{\partial x} = 2x \;\&\; \frac{\partial M}{\partial y} = x+2y$$

x varies from 0 to 1 and y varies from x^2 to x.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_0^1 \int_{x^2}^x (2x - x - 2y)dydx$$

$$= \int_0^1 \int_{x^2}^x (x-2y)dydx = \int_0^1 \left(xy - 2\frac{y^2}{2}\right)_{x^2}^x dx$$

$$= \int_0^1 [(xx - x^2) - (x^3 - x^4)]dx = \int_0^1 (x^4 - x^3)dx$$

$= \left(\frac{x^5}{5} - \frac{x^4}{4}\right)_0^1 = \frac{1}{5} - \frac{1}{4} = \frac{4-5}{20} = -\frac{1}{20}$ .... (2)

From (1) and (2)

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

Hence, Green's theorem is verified.

Problem 2 : Verify Green's theorem in plane for the integral $\int_C x^2dx + ydy$, where C is the curve enclosing the region R bounded by the parabola $y^2 = x$ and the line $y = x$.

(Hint : Common point (0,0) , (1,1). For the line segment x=t, y=t & t varies from 0 to 1. For the parabolic arc $x = t^2$ & $y = t$, where t varies from 1 to 0. Ans. -1/28).

Problem 3 : Verify Green's theorem in plane for the integral $\int_C x^2dx + xydy$, where C is the curve enclosing the region R bounded by the parabola $y^2 = 8x$ and the line $y = 2x$.

(Hint : Common point (0,0) , (2,4). For the line segment x=t, y=2t & t varies from 0 to 2. For the parabolic arc $x = 2t^2$ & $y = 4t$, where t varies from 1 to 0. Ans. 8/3)

Problem 4 : Verify Green's theorem for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$, where C is the boundary of the region R enclosed by $y = x^2$ & $y^2 = x$.

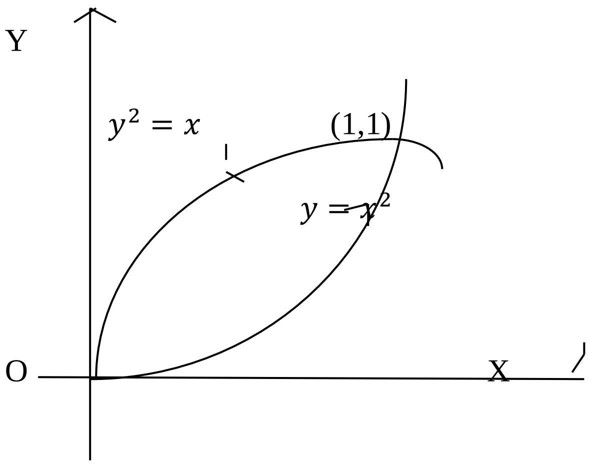
Solution: Given parabolas are $y = x^2$ & $y^2 = x$.

$\therefore x^4 = x => x^4 - x = 0 => x(x^3 - 1) = 0 => x = 0 \; or \; x = 1.$

When x=0 , y=0.

When x=1 , y=1.

Let the parabolas intersect at (0,0) and (1,1).



Now, the curve C composed of the arc Γ of the parabola $y = x^2$ and the arc Γ' of the parabola $y^2 = x$.

The parametric equation of Γ is $x = t, y = t^2$, where t varies from 0 to 1.

$\int_\Gamma = \int_0^1 (3t^2 - 8t^4)dt + (4t^2 - 6t^3)(2tdt) = -1 \; (verify)$

The parametric equation of Γ' is $x = t^2$ & $y = t$, where t varies from 1 to 0.

$\int_{\Gamma'} = \int_1^0 (3t^4 - 8t^2)(2tdt) + (4t - 6t^3)(dt) = \frac{5}{2} \; (verify)$

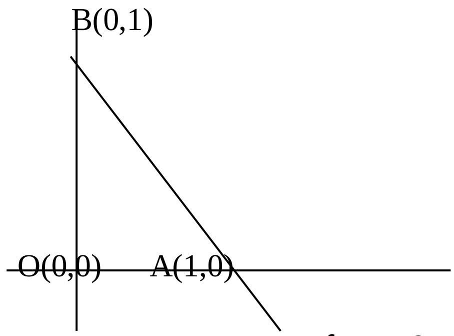
Thus $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy = -1 + \frac{5}{2} = \frac{3}{2} \ldots\ldots\ldots\ldots\ldots\ldots.(1)$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_0^1 \int_{x^2}^{\sqrt{x}} 10ydydx = \frac{3}{2}(verify) \ldots\ldots..(2)$$

From (1) & (2) Green's theorem is verified.

Problem 4 Verify Green's theorem for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy,$ where C is the boundary of the region R enclosed by x=0, y=0, x+y=1.

Solution :



$\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy = \int_{OA} (3x^2 - 8y^2)dx + (4y - 6xy)dy +$
$\int_{AB} (3x^2 - 8y^2)dx + (4y - 6xy)dy + \int_{BO} (3x^2 - 8y^2)dx + (4y - 6xy)dy$

Along OA : x=t, y=0, t varies from 0 to 1.

$$\int_{OA} (3x^2 - 8y^2)dx + (4y - 6xy)dy = \int_0^1 3t^2 dt = 1\ (verify)$$

Along AB :

$\frac{x-1}{0-1} = \frac{y-0}{1-0} = t \Rightarrow \frac{x-1}{-1} = t \Rightarrow x - 1 = -t \Rightarrow x = 1 - t \,\&\, \frac{y}{1} = t \Rightarrow y = t$

$x = 1 - t, y = t,$ t varies from 0 to 1.

$$\int_{AB} (3x^2 - 8y^2)dx + (4y - 6xy)dy = \int_0^1 (-3 + 4t + 11t^2)dt = 8/3\ (verify)$$

Along BO : $x = 0, y = 1 - t,$ t varies from 0 to 1.

$$\int_{BO} (3x^2 - 8y^2)dx + (4y - 6xy)dy = \int_0^1 4(t - 1)dt = -2\ (verify)$$

Thus, $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy = 1 + \frac{8}{3} - 2 = \frac{5}{3}(verify)$ .... (1)

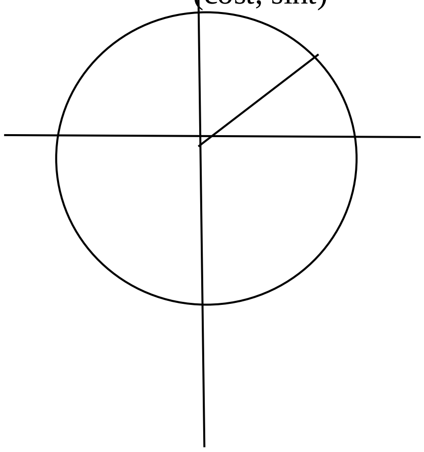
Find $\frac{\partial N}{\partial x} \,\&\, \frac{\partial M}{\partial y}$

Then, $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_0^1 \int_0^{1-x} (-6y + 16y)dydx = \frac{5}{3}(verify)$.... (2)

From (1) & (2),

Problem 5 : Verify Green's theorem for $\int_C (x-2y)dx + xdy$, where C is the circle $x^2 + y^2 = 1$.

Solution :



The parametric equations of the circle are x=cost, y=sint, t varies from 0 to 2π.

$$dx = -sint\ dt \ \&\ dy \ = \ cost\ dt$$

$$\int_C (x-2y)dx + xdy = \int_0^{2\pi} (cost - 2sint)(-sintdt) + costcostdt$$

$$= \int_0^{2\pi} (-costsint + 2sin^2t + cos^2t)dy$$

$$= \int_0^{2\pi} \left(-\frac{sin2t}{2} + 2sin^2t + cos^2t\right) dy = 3\pi\ (verifty)$$

Problem 6 : Evaluate $\int_C (3x+4y)dx + (2x-3y)dy$, where C is the circle $x^2 + y^2 = 4$

Problem 7 : Show that $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy = \frac{5}{3}$, where C is the boundary of the rectangular area enclosed by the lines y=0, x+y=1, x=0.

Problem 8 : Show that $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy = 20$, where C is the boundary of the rectangular area enclosed by the lines x=0, x=1, y=0, y=2.

Problem 9 : Evaluate $\int_C xy^2dy - x^2ydx$, where C is the cardioids $r = a(1 + cos\theta)$.

[Hint:

$\int_C xy^2dx - x^2ydy = \iint_R (x^2 + y^2)dxdy = \int_0^{2\pi} \int_0^{a(1+cos\theta)} r^2(rdr)d\theta \ = \frac{35}{16}\pi a^4$]

4.2 STOKE'S THEOREM

Theorem : If S is an open two sided surface bounded by a simple closed curve C and f is a vector valued function having continuous first order partial derivatives then

$$\int_C f \cdot dr = \iint_S (\nabla \times f) \cdot n \, dS$$

where C is traversed in the anticlockwise direction.

Problem 10: Verify Stokes theorem for the vector function $f = y^2 i + yj - xzk$ and S is the upper half of the sphere $x^2 + y^2 + z^2 = a^2 \; and \; z \geq 0$.

Solution : By problem 2 of 3.2 $\iint_S (\nabla \times f) \cdot n \, dS = 0$

Now the boundary C of the hemisphere is given by the equation $x = a\,cos\theta, y = a\,sin\theta, z = 0, 0 \leq \theta \leq 2\pi$.

$$\int_C f \cdot dr = \int_C y^2\,dx + ydy - xzdz = \int_0^{2\pi} [a^2 sin^2\theta(-asin\theta) + asin\theta(acos\theta)]d\theta$$

$$= -a^3 \int_0^{2\pi} sin^3\theta d\theta + a^2 \int_0^{2\pi} sin\theta\,cos\theta d\theta = 0$$

Thus, $\int_C f \cdot dr = \iint_S (\nabla \times f) \cdot n \, dS$.

Hence Stoke's theorem is verified.

Problem 11 : By using Stoke's theorem prove that $\int_C r \cdot dr = 0 \; where \; r = xi + yj + zk$.

Solution : $\nabla \times r = 0$.

By Stoke's theorem we have, $\int_C r \cdot dr = \iint_S (\nabla \times r) \cdot n \, dS$=0.

Problem 12 : Evaluate by using Stoke's theorem $\int_C (yzdx + zxdy + xydz)$ where C is the curve $x^2 + y^2 = 1\,, z = y^2$.

Solution : We note that $yzdx + zxdy + xydz = (yzi + zxj +$ xyk)$\cdot$ $(idx + jdy + kdz)$

$$= f \cdot dr \; where \; f = yzi + zxj + xyk \; and \; dr = idx + jdy + kdz$$

Now, $\int_C (yzdx + zxdy + xydz) = \int_C f \cdot dr = 0 = \iint_S (\nabla \times f) \cdot n \, dS$

But $\nabla \times f = 0$

$$\therefore \int_C (yzdx + zxdy + xydz) = 0.$$

Problem 13 : Evaluate $\int_C e^x dx + 2ydy - dz \ \ by$ using Stoke's theorem where C is the curve $x^2 + y^2 = 4\,, z = 2.$

Solution : $e^x dx + 2ydy - dz = (e^x i + 2yj - k) \cdot (idx + jdy + kdz) where$

$$f = (e^x i + 2yj - k) \ and \ dr = (idx + jdy + kdz)$$

$\int_C e^x dx + 2ydy - dz = \int_C f \cdot dr = 0 = \iint_S (\nabla \times f) \cdot n\, dS$ where S is any surface whose boundary is given by $x^2 + y^2 = 4\,, z = 2.$

Now, $\nabla \times f = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x & 2y & -1 \end{vmatrix} = 0$

$$\therefore \iint_S (\nabla \times f) \cdot n\, dS = 0.$$

$$\therefore \int_C e^x dx + 2ydy - dz = 0.$$

4.3 Gauss Divergence theorem

If V is the volume bounded by a closed surface S and f is a vector valued function having continuous partial derivatives then $\iint_S f \cdot ndS = \iiint_V \nabla \cdot f dV.$

Problem 14 : Verify Gauss divergence theorem for the vector function $f = (x^3 - yz)i - 2x^2yj + 2k$ over the cube bounded by $x = 0, y = 0, z = 0, x = a, y = a \ and \ z = a.$

Solution : By problem 3 of 3.2 we proved that $\iint_S f \cdot n\, dS = \frac{a^5}{3}.$

Now, $\nabla \cdot f = 3x^2 - 2x^2 = x^2.$

$$\iiint_V \nabla \cdot f \ dV = \int_0^a \int_0^a \int_0^a x^2 dx\, dy\, dz = \frac{1}{3}\int_0^a \int_0^a a^3\, dy\, dz = \frac{1}{3}\int_0^a a^4\, dz = \frac{a^5}{3}.$$

$$\therefore \iint_S f \cdot ndS = \iiint_V \nabla \cdot f dx\, dy\, dz.$$

Hence Gauss divergence theorem is verified.

Problem 15 : Verify Gauss divergence theorem for $f = yi + xj + z^2k$ for the cylindrical region S given by $x^2 + y^2 = a^2; z = 0 \; and \; z = h$.

Solution : $\nabla \cdot f = 2z$

$$\therefore \iiint_V \nabla \cdot f \; dV = \int_0^h \int_0^{2\pi} \int_0^h 2zr \, dr \, d\theta \, dz \; (changing \; into \; cylindrical \; coordinates)$$

$$= \int_0^h \int_0^{2\pi} a^2 z \, d\theta \, dz = \int_0^h 2a^2 \pi z \, dz = \pi a^2 h^2.$$

The surface S of the cylinder consists of a base S_1, the top S_2 and the curved portion S_3.

On $S_1, z = 0, n = -k. \; Hence \; f \cdot n = 0.$

$$\therefore \iint_{S_1} f \cdot n \, dS = 0.$$

On $S_2, z = h, n = k. \; Hence \; f \cdot n = h^2.$

$$\therefore \iint_{S_2} f \cdot n \, dS = \iint_{S_2} h^2 dxdy \; where \; D \; is \; the \; region \; bounded \; by \; the \; circle$$

$$x^2 + y^2 = a^2.$$

$$= \pi h^2 a^2.$$

On $S_3, n = \frac{\nabla\emptyset}{|\nabla\emptyset|} \; where \; \emptyset = x^2 + y^2 - a^2 = \frac{2xi+2yj}{2\sqrt{x^2+y^2}} = \frac{xi+yj}{a}.$

$$n \cdot j = \frac{y}{a}.$$

$$\frac{f \cdot n}{|n \cdot j|} = 2x.$$

$$\iint_{S_3} f \cdot n \, dS = \iint_R 2xdy \, dz = a^2 \int_0^b \int_0^{2\pi} 2 \, cos\theta \, d\theta \, dz = 0.$$

$\therefore \iint_S f \cdot n \, dS = \iint_{S_1} f \cdot n \, dS + \iint_{S_2} f \cdot n \, dS + \iint_{S_3} f \cdot n \, dS = 0 + \pi h^2 a^2 + 0 = \pi h^2 a^2.$

$\therefore \iiint_V \nabla \cdot f \; dV = \iint_S f \cdot n \, dS = \pi h^2 a^2.$

Hence Gauss divergence theorem is verified.

Problem 16 : Prove that for a closed surface S, $\iint_S r \cdot n\, dS = 3V$, where V is the volume enclosed by S.

Solution : By Gauss' divergence theorem we have,

$$\iint_S r \cdot n\, dS = \iiint_V \nabla \cdot r\ dV$$

$$= 3 \iiint_V dV = 3V \textit{ where } V \textit{ is the volume enclosed by } S.$$

Problem 17 : Show that $\iint_S f \cdot n\, dS = \iiint_V a^2\ dV\ \ where\, r = \emptyset a\ and\ a = \nabla\emptyset and\ \nabla^2\emptyset = 0.$

UNIT V

FOURIER SERIES

5.0 Introduction. Fourier series named after the French Mathematician cum Physicist Jean Baptiste Joseph Fourier (1768-1803), has several interesting applications in engineering problems. He introduced Fourier series in 1822 while he was investigating the problem of heat conduction. This series became a very important tool in mathematics. In this chapter we discuss the basic concepts relating to Fourier series development of several functions.

5.1 Periodic Functions. A function $f: \mathbb{R} \to \mathbb{R}$ is said to be **periodic** if there exists a positive numberωsuch that$f(x + \omega) = f(x)$for all real numbers x and ω is called a **period** of$f$. If a periodic function has asmallest positive period ω, then $\omega$is called the **primitive period** of$f$.

Example 1. The *trigonometric functions*$\sin x$ and $\cos x$ are periodicfunctions with primitive period 2π [since $sin\,(x + 2\pi) = \sin x$ and$cos\,(x + 2\pi) = \cos x$].

Example 2.$\sin 2x$ and $\cos 2x$ are periodic functions with primitive periodπeach.

Example 3. The *constant function*$f(x) = c$ is a periodic function.In fact,every positive real number is a period of$f$ and hence this periodic functionhas no primitive period.

Example 4. Let $f: \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 0 & if\ x\ is\ rational \\ 1 & if\ x\ is\ irrational \end{cases}$$

Let ωbe any rational number. If x is rational then $x + \omega$ is also rational andif$x$ is irrational then$x + \omega$ is also irrational. Hence,

$$f(x + \omega) = \begin{cases} 0 & if\ x\ is\ rational \\ 1 & if\ x\ is\ irrational \end{cases}$$

$$= f(x)$$

Hence every rational number is a period of $f$and$f$ has no primitive period.

Remark. Let$f$be a periodic function with periodω. If the values of$f(x)$are known in an interval of lengthω, then by periodicity $f(x)$ can bedetermined for all x.Hence the graph of a periodic function is obtained byperiodic function of its graph in any interval of length ω.

Example 1. The graph of the periodic function$f(x) = \sin x$ is given below in

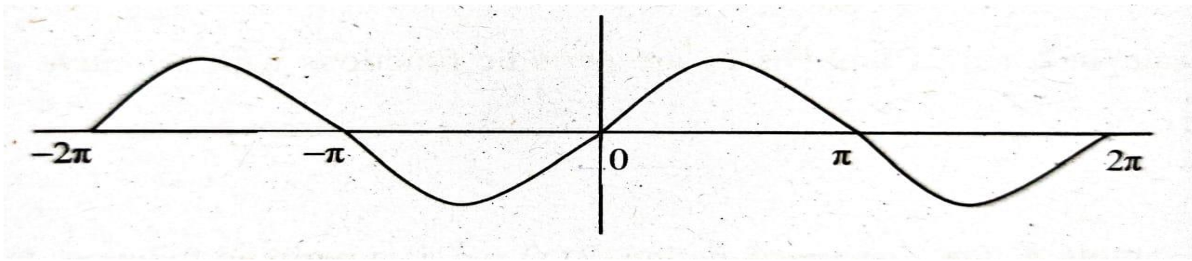


Figure 1

Example 2. Let f be the periodic function defined by

$$f(x) = \begin{cases} -1 \; if -\pi \leq x < 0 \\ \;\; 1 \; if \quad 0 \leq x < \pi \end{cases} \; and \; f(x + 2\pi) = f(x).$$

The graph of the periodic function $\sin x$ is given below in figure 2.

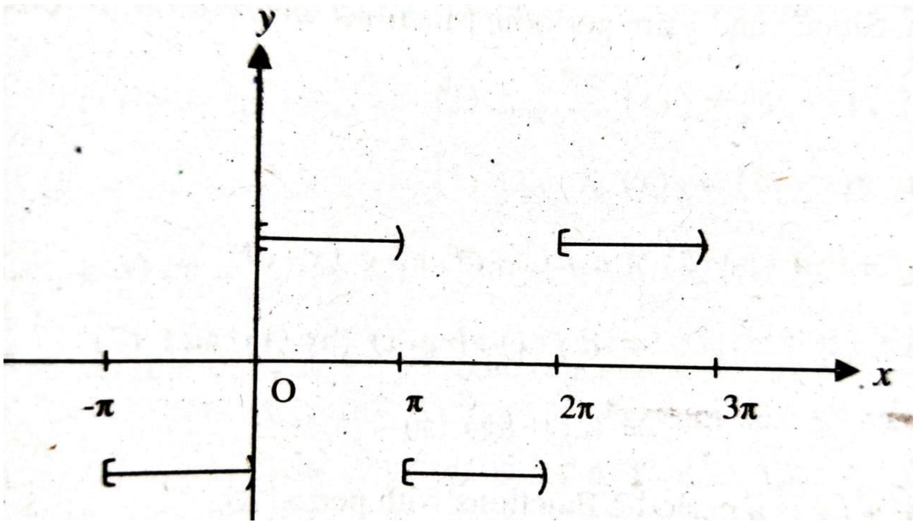


Figure 2

Example 3. Let f be a periodic function defined by

$$f(x) = \begin{cases} x \; if -\pi \leq x < \pi \\ f(x + 2\pi) = f(x) \end{cases} \; and$$

(ie) $f(x) = x$ if $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$

The graph of the periodic function $\sin x$ is given below in figure 3.

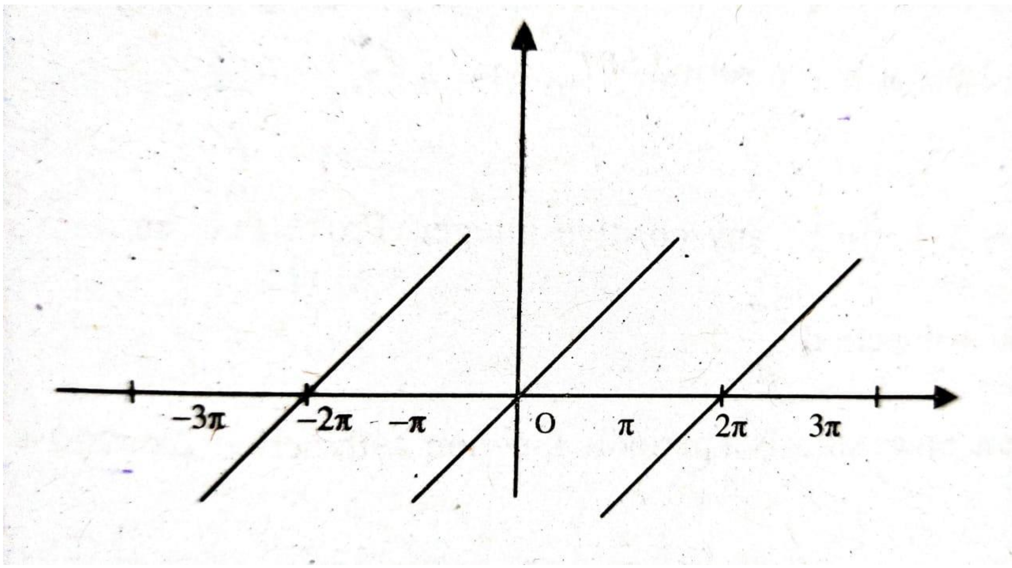


Figure 3

Solved Problems.

Problem 1. Let f and g be periodic functions with period ϖ each and let a and b be real numbers. Prove that $af + bg$ is also a periodic function with period ϖ.

Solution. Since f and g are periodic functions with period ϖ each we have for all x,

$$f(x+\varpi) = f(x) \ldots\ldots\ldots\ldots\ldots.(1)\ and$$
$$g(x+\varpi) = g(x) \ldots\ldots\ldots\ldots\ldots(2)$$

Now, $(af + bg)(x+\varpi) = af(x+\varpi) + bg(x+\varpi)$

$$= af(x) + bg(x)\ (by\ (1) and\ (2))$$

$$= (af + bg)(x)$$

Hence $af + bg$ is a periodic function with period ϖ.

Problem 2. If ϖ is a period of f'prove that $n\varpi$ is also a period of f where n is any positive integer.

Solution. Let n be any positive integer. Since ϖ is a period of f we have $f(x) = f(x + \omega)$. Using this fact repeatedly we have

$$f(x) = f(x+\omega) = f(x+2\omega) = \cdots = f(x+(n-1)\omega) = f(x+n\omega)$$

It follows that $n\ \omega$ is a period of f.

Problem 3. Let n be any positive integer. Prove that $\sin nx$ is a periodicfunction with period $\frac{2\pi}{n}$.

Solution. Since $\sin x$ is a periodic function with period 2π we have $\sin(x+2\pi)=\sin x$ for all x.

Now, let $g(x)=\sin x$

Then $g\left(x+\frac{2\pi}{n}\right)=\sin\left[n\left(x+\frac{2\pi}{n}\right)\right]=\sin(nx+2\pi)=\sin nx=g(x)$

Hence $\sin nx$ is a periodic function with period $\frac{2\pi}{n}$.

Problem 4. Let $f(x)$ be a periodic function with period ϖ. Prove that forany positive real number$a$,$f(ax)$is aperiodic function with period$\frac{\varpi}{a}$.

Solution. Since$f(x)$is a periodicfunction with period 2π we have$f(x+\omega)=f(x)$for all$x$. Let$g(x)=f(ax)$.

Now, $g\left(x+\frac{\varpi}{a}\right)=f\left[a\left(x+\frac{\varpi}{a}\right)\right]=f(ax+\varpi)=f(ax)=g(x)$.

Hence $g(x)$ is a periodic function with period$\frac{\varpi}{a}$.

Exercises.

1. Find the primitive period of the following functions
   (a)$\sin 2x$(b)$\cos 2x$(c)$\cos nx$(d)$\sin \pi x$ (e) $\cos 2\pi\, x$ (f)$\cos\left[\left(\frac{2\pi\, x}{k}\right)\right]$

Answers. 1. (a) π (b) π (c) $\frac{2\pi}{n}$ (d)2 (e)1 (f)k

5.2 FOURIER SERIES - FULL RANGE

Since periodic functions which occur frequently in engineering problems are rathercomplicated, representation of periodic functions interms of a simple periodic function is a matter of great practical importance.We nowdiscuss the problem of representing various functions of period2π.(**Full range**) in terms of the simple functions namely constant function$c$ and some trigonometric functions $\sin x, \cos x, \sin 2x, \cos 2x, \ldots, \sin nx, \cos nx$etc.

Definition. Trigonometric Series. A series of the form

$$a_0+a_1\cos x+b_1\sin x+a_2\cos 2x+b_2\sin 2x+\cdots+a_n\cos nx+b_n\sin nx+\cdots$$
$$=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

Wherea_n and $b_n$are real constants is called a **trigonometric series**a_n and b_n are called the coefficients of the series (**Fourier coefficients**). Since each term of the trigonometric series is a function of period 2π it follows that if the series converges then the sum is also a function of period 2π.

We now state the following theorem and its results without proof and it becomes the definition of **Fourier series**

$$a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots + a_n \cos nx + b_n \sin nx$$

Theorem 1. Let $f(x)$ be a periodic function with period 2π. Suppose $f(x)$ can be represented as a **trigonometric series.**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

Then we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx \qquad n = 1,2,3,\ldots$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx \qquad n = 1,2,3,\ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx \qquad n = 1,2,3,\ldots$$

Remark 1. The formulae for the coefficientsa_0, a_n, b_n, given in the above theorem are known as Fourier coefficients.

Euler's Formulae.

Definition - Fourier series

The series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$when a_0, a_n, b_n are given by the *Euler's formulae*is calledthe Fourier series of$f(x)$.Also, thecoefficients a_0, a_n, b_n are called Fourier coefficients.

Remark 2. We use$\frac{a_0}{2}$ instead of a_0 in the Fourier series just to obtain uniformity in *Euler's formulae.*

Remark 3. If$f(x)$ is a periodic function with period 2π we can obtain the Fourier series of $f(x)$in any interval of length 2π. If the interval is taken as $(c, c + 2\pi)$then the Euler's Formulae for Fourier coefficients are given by

$$a_0 = \frac{1}{\pi}\int_0^{c+2\pi} f(x)\ dx$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\ dx \qquad n = 1,2,3,\dots$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\ dx \qquad n = 1,2,3,\dots$$

The calculation of the*Fourier coefficients* of a function can be simplified for certain functions.

Definition. Areal function $f(x)$ is called an **even function** if$f(-x) = f(x)$for all x.

The function$f(x)$ is called an **odd function** if$f(-x) = -f(x)$.

For example, (i) $\cos nx$ is an even function, (ii) $\sin nx$ is an odd function.(iii) x^n is an odd function if$n$ is an odd integer and an even function if$n$an even integer.

Remark 1. If$f(x)$ is an even function $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

If$f(x)$ is an odd function $\int_{-a}^{a} f(x)\,dx = 0$

Remark 2.

(i) The product of two even functions is an even function.
(ii) The product of two odd functions is an even function.
(iii)The product of an even function and an odd function is anodd function.

Remark 3. If $f(x)$ is. an odd function then $f(x)\cos nx$is also an odd function.

Hence $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx = 0\ and\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx = 0$

Thus, for an odd function the Fourier coefficientsa_0 *and* a_n*are* 0.

Also $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\ dx$

(Since,$f(x)\sin nx$ is an even function.)

Remark 4. If $f(x)$ isan even function then $f(x)\sin nx$ is an oddfunction.

Hence $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx = 0$for all$n$.

Using the above remarks, we give below the **working rules** for calculatingthe Fourier coefficientsof a periodic function with period 2π.

Working Rules.

Let $f(x)$ be a periodic function with period2π.Suppose the given interval is $(-\pi,\pi)$.

Step 1. Check whether $f(x)$ is an even function or an odd function.

Step 2. (i) If $f(x)$ is aneven function then $b_n = 0$ for all n and

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \ dx \ for \ all \ n \geq 0$$

(ii) If $f(x)$ is an odd function then $a_n = 0$ for all $n \geq 0$ and

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \ dx$$

Step 3. If$f(x)$ is neither an even function nor an odd function in$(-\pi,\pi)$or if the given interval is not$(-\pi,\pi)$, then calculate the Fourier coefficientsby using Euler's formulae (refer Remark I)

The following results on integration will be useful in calculating the Fouriercoefficients.

Result 1. Bernoulli's formula.

$$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots \qquad where$$

$$u' = \frac{du}{dx}, u'' = \frac{d^2u}{dx^2}, u''' = \frac{d^3u}{dx^3} \ etc$$

$$and \ v_1 = \int v\,dx, v_2 = \int v_1\,dx, v_3 = \int v_2\,dx, \ldots . etc$$

Result 2. (i) $\int e^{ax}\sin bx \ dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$

(ii) $\int e^{ax}\cos bx \ dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx - b\sin bx]$

Result 3. Noe let $g(x) = \sin nx$

Then $g\left(x + \frac{2\pi}{n}\right) = \sin\left[n\left(x + \frac{2\pi}{n}\right)\right] = \sin(nx + 2\pi) = \sin nx = g(x)$

Hence $g(x) = \sin nx$ is a periodic function with period $\frac{2\pi}{n}$ where n is a positive integer.

Solved Problems.

Problem 1. Determine the Fourier expansion of the function $f(x) = x$ where $-\pi \leq x \leq \pi$

Solution. Obviously $f(x) = x$ is an odd function

Hence $a_n = 0$ for all $n \geq 0$.

Now, $b_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \ dx = \frac{2}{\pi}\int_{-\pi}^{\pi} x \sin nx \ dx$

Taking $u = x$ and $dv = \sin nx \ dx$ and applying Bernoulli's formula we get

$$b_n = \frac{2}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= -\frac{2}{n\pi}[\pi \cos n\pi]$$

$$= \frac{-2(-1)^n}{n}$$

$$= \frac{2(-1)^{n+1}}{n}$$

Hence $x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin nx$

$$\therefore x = 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right].$$

Problem 2. Find the Fourier series for the function $f(x) = x^2$ where $-\pi \leq x \leq \pi$ and deduce that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} - \cdots = \frac{\pi^2}{8}$

Solution. Let $f(x) = x^2$. We note that is an even function.

$$\text{Hence, } a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \ dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \ dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \ dx \ (\because x^2 \ is \ an \ even \ function)$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2\pi^2}{3}$$

Where, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \ dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \ dx$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \; dx \; (\because x^2 \cos nx \; is \; an \; even \; function)$$

Now by applying the Bernoulli's formula

$$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$$

Where $u = x^2$and $dv = \cos nx \; dx$ so that $u' = 2x, ; u'' = 2; u''' = 0$

Now,$a_n = \frac{2}{\pi}\left[\frac{2\pi \cos nx}{n^2}\right]_0^{\pi} = \frac{4(-1)^n}{n^2}$

Now, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \; dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx \; dx = 0$ (since $x^2 \sin nx$ is an odd function).

Hence, $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\left(\frac{(-1)^n \cos nx}{n^2}\right)$………………… (1)

Deduction. (i) Put $x = \pi$ in (1) and we get

$$\pi^2 = \frac{\pi^2}{3} + 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots\right).$$

Hence, $4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots\right) = \pi^2 - \frac{\pi^2}{3} = \frac{2\pi^2}{3}$.

Hence, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$.

(ii) Put $x = 0$ in (1) and we get

$$\therefore 0 = \frac{\pi^2}{3} + 4\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots\right)$$

(i) Hence, we get $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$

Adding the results (i) and (ii) we get

$$2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{n^2} + \cdots\right) = \frac{\pi^2}{4}$$

Hence, $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Problem 3. Show that in the range 0 to 2π the Fourier series expansion for e^x is

$$\frac{e^{2\pi} - 1}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2 + 1}\right) - \sum_{n=1}^{\infty}\left(\frac{\sin nx}{n^2 + 1}\right)\right]$$

Solution. Let $f(x) = e^x$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\ dx = \frac{1}{\pi}\int_0^{2\pi} e^x\ dx = \frac{1}{\pi}[e^x]_0^{2\pi} = \frac{e^{2\pi}-1}{\pi}.$$

Now taking, $I_n = \frac{1}{\pi}\int_0^{2\pi} e^x \cos nx\ dx$………………. (1)

$$= [e^x \cos nx]_0^{2\pi} + n\int_0^{2\pi} e^x \sin nx\ dx$$

$$= (e^{2\pi}-1) + n\left[\{e^x \sin nx\}_0^{2\pi} - n\int_0^{2\pi} e^x \cos nx\ dx\right]$$

$$\therefore I_n = (e^{2\pi}-1) - n^2 I_n$$

$$\therefore (n^2+1)I_n = (e^{2\pi}-1)$$

$$\therefore I_n = \left(\frac{e^{2\pi}-1}{n^2+1}\right)$$

$$Hence, a_n = \frac{1}{\pi}\left(\frac{e^{2\pi}-1}{n^2+1}\right)$$

Similarly, we can prove that $b_n = -\left(\frac{n(e^{2\pi}-1)}{\pi(n^2+1)}\right)$ (verify)

$$\therefore e^x = \frac{e^{2\pi}-1}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2+1}\right) - \sum_{n=1}^{\infty}\left(\frac{\sin nx}{n^2+1}\right)\right]$$

Problem 4. If $f(x) = \begin{cases} -x & if -\pi < x < 0 \\ x & if \quad 0 < x < \pi \end{cases}$ expand $f(x)$ as a Fourier series in the interval $(-\pi, \pi)$.

Solution. Clearly $f(-x) = f(x)$ for all $x \in (-\pi, \pi)$. Hence $f(x)$ is an even function in $(-\pi, \pi)$. Hence the function can be expanded as a Fourier series of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx$.

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx = \frac{2}{\pi}\int_0^{\pi} f(x)\ dx$ (Since $f(x)$ is an even function)

$$= \frac{2}{\pi}\int_0^{\pi} x\ dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{x^2}{2}\right] = \pi$$

Now, $a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx\ dx$

$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \frac{2}{n\pi}\int_0^{\pi} \sin nx \; dx = \frac{2}{\pi n^2}[\cos nx]_0^{\pi}$$

$$= \frac{2}{\pi n^2}[(-1)^n - 1]$$

$$= \begin{cases} -\frac{4}{\pi n^2} & if\ n\ is\ odd \\ 0 & if\ n\ is\ even \end{cases}$$

Hence, $f(x) = \frac{\pi}{2} - \frac{\pi}{4}\sum\left(\frac{\cos nx}{n^2}\right)$ where n is odd.

$$\therefore f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

Note. This problem can also be re stated as $f(x) = |x|$ in the interval $-\pi < x < \pi$.

Problem 5. Find the Fourier series of the function $f(x) = \begin{cases} \pi + 2x & if \ -\pi < x < 0 \\ \pi - 2x & if \ \ 0 \leq x < \pi \end{cases}$

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

Solution. Here the given function is $f(x) = \pi - 2|x|$ and hence it is an even function. Hence $b_n = 0$ for all n.

Now, $a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi - 2x)\,dx = -\frac{1}{\pi}[(\pi - 2x)^2]_0^{\pi} = -\frac{1}{\pi}[(-\pi)^2 - \pi^2] = 0$

Also, $a_n = \frac{2}{\pi}\int_0^{\pi}(\pi - 2x)\cos nx\,dx$

$$= \frac{2}{\pi}\left[(\pi - 2x)\frac{\sin nx}{n}\right]_0^{\pi} + \frac{4}{\pi}\int_0^{\pi} \sin nx\,dx$$

$$= 0 + \frac{4}{\pi n^2}[-\cos nx]_0^{\pi}$$

$$= \frac{4}{\pi n^2}[-\cos nx]_0^{\pi}$$

$$= \frac{4}{\pi n^2}[-(-1)^n + 1]$$

$$= \begin{cases} 0 & if \ \ n\ is\ odd \\ \frac{8}{\pi n^2} & if\ n\ is\ even \end{cases}$$

$$\therefore f(x) = \frac{8}{\pi}\sum\left(\frac{\cos(2n-1)\pi}{(2n-1)^2}\right)$$

Putting $x = 0$ in the above result we get $f(0) = \frac{8}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$

$$\therefore\ \pi = \frac{8}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]\ (since\ f(0) = \pi, by\ definition)$$

$$Hence, \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

Problem 6. Find the Fourier series for $f(x) = |\sin x|$ in $(-\pi, \pi)$ of periodicity 2π.

Solution. We note that $f(x) = |\sin x|$ is an even function of x through $\sin x$ is an odd function. Hence $f(x)$ will contain only cosine terms in its Fourier series.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x|\ dx$

$= \frac{2}{\pi}\int_0^{\pi} |\sin x|\ dx$ (Since $|\sin x|$ is an even function)

$$= \frac{2}{\pi}\int_0^{\pi} \sin x\ dx$$

$$= \frac{2}{\pi}[-\cos x]_0^{\pi}$$

$$= \frac{2}{\pi}[1 + 1]$$

$$= \frac{4}{\pi}$$

Now, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x| \cos nx\ dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} |\sin x| \cos nx\ dx + \frac{1}{\pi}\int_{0}^{\pi} |\sin x| \cos nx\ dx$$

$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\ dx$ ($\because f(x) = |\sin x| = \sin x\ in\ [-\pi, \pi]$)

$$= \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]\, dx$$

$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n+1)x}{n+1}\right]_0^{\pi}$ if $n \neq 1$

$$= \frac{1}{\pi}\left[-\frac{1}{n+1}|\{(-1)^{n+1} - 1\}| + \frac{1}{n-1}\{(-1)^{n-1} - 1\}\right]$$

$$=\frac{1}{\pi}\left[-\frac{1}{n+1}\{1+(-1)^n\}+\frac{1}{n-1}\{1+(-1)^n\}\right]$$

$$=\frac{1}{\pi}\left(\frac{-2}{n^2-1}\right)\{1+(-1)^n\}$$

$$\therefore For\ n>1, a_n=f(x)=\begin{cases}0 & if\ n\ is\ odd\\ -\dfrac{4}{\pi(n^2-1)} & if\ n\ is\ even\ and\ n\ \neq 1\end{cases}$$

If $n=1$, $a_1=\frac{2}{\pi}\int_0^{\pi}\sin x\cos x\ dx=\frac{2}{\pi}\int_0^{\pi}\sin x\ d(\sin x)=\frac{2}{\pi}\left[\frac{sin^2x}{2}\right]_0^{\pi}=0.$

$$\therefore f(x)=|\sin x|=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{(4n^2-1)}$$

$$=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{(2n-1)(2n+1)}$$

$$\therefore |\sin x|=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{\cos 2nx}{(2n-1)(2n+1)}\right]$$

Problem 7. If $f(x)=\begin{cases}-\frac{\pi}{4} & if -\pi<x<0\\ \frac{\pi}{4} & if\quad 0<x<\pi\end{cases}.$

Solution. We note that $f(x)$ is a periodic function with period 2π.

Now, $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\ dx=\frac{1}{\pi}\int_{-\pi}^{0}\left(-\frac{\pi}{4}\right)\ dx+\frac{1}{\pi}\int_0^{\pi}\left(\frac{\pi}{4}\right)\ dx$

$$=\frac{1}{\pi}\left[\left(-\frac{\pi}{4}\right)[x]_{-\pi}^{0}\right]+\frac{1}{\pi}\left[\left(\frac{\pi}{4}\right)[x]_0^{\pi}\right]$$

$$=-\frac{1}{\pi}\left(\frac{\pi}{4}\right)(\pi)+\frac{1}{\pi}\left(\frac{\pi}{4}\right)(\pi)$$

$$=-\frac{\pi}{4}+\frac{\pi}{4}=0$$

Now, $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\ dx=\int_{-\pi}^{0}\left(-\frac{\pi}{4}\right)\cos nx\ dx+\frac{1}{\pi}\int_0^{\pi}\left(\frac{\pi}{4}\right)\cos nx\ dx$

$$=-\frac{\pi}{4}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0}+\frac{\pi}{4}\left[\frac{\sin nx}{n}\right]_0^{\pi}$$

$$=-\frac{\pi}{4}[0-0]+\frac{\pi}{4}[0-0]=0$$

Now, for $n>1$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx = \int_{-\pi}^{0}\left(-\frac{\pi}{4}\right)\sin nx\ dx + \int_{0}^{\pi}\left(\frac{\pi}{4}\right)\sin nx\ dx$$

Hence, $b_n = -\frac{\pi}{4}\int_{-\pi}^{0}\sin nx\ dx + \frac{\pi}{4}\int_{0}^{\pi}\sin nx\ dx$

$$= 0 - \frac{\pi}{4}\left[\frac{\cos nx}{n}\right]_0^{\pi} = -\frac{\pi}{4n}[\cos n\pi - 1]$$

Thus $b_n = -\frac{\pi}{4n}[(-1)^n - 1]$

Hence $b_1 = -\frac{\pi}{4}[-1-1] = \frac{\pi}{2}$

$$b_2 = -\frac{\pi}{8}[1-1] = 0$$

$$b_3 = -\frac{\pi}{12}[-2] = \frac{\pi}{6}$$

$b_4 = -\frac{\pi}{16}[1-1] = 0$ etc

Hence $f(x) = \frac{\pi}{2}\sin x + \frac{\pi}{6}\sin 3x + \cdots$

(ie) $f(x) = \frac{\pi}{2}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$

Problem 8. Find the Fourier series for defined in $f(x) = e^x$ defined in $[-\pi, \pi]$

Solution. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\ dx = \frac{1}{\pi}[e^x]_{-\pi}^{\pi} = \frac{1}{\pi}(e^{\pi} - e^{-\pi}) = \frac{2\sinh\pi}{\pi}$.

Now, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx = \frac{1}{\pi}\int_{0}^{2\pi} e^x\cos nx\ dx$

$$= \frac{1}{\pi}\left[\frac{e^x}{1^2 + n^2}(\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

[using the formula in integration $\int e^{ax}\cos bx\ dx = \frac{e^x}{a^2+b^2}(a\cos bx + b\sin bx)$]

$$\therefore a_n = \frac{1}{\pi(n^2+1)}[e^{\pi}\cos n\pi - e^{-\pi}\cos n\pi]$$

$$= \frac{\cos n\pi\,(e\pi - e^{-\pi})}{\pi(n^2+1)} = \frac{(-1)^n 2\sinh\pi}{\pi(n^2+1)}$$

Now, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\sin nx\ dx$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{1^2 + n^2}(\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

[using the formula $\int e^{ax}\cos bx\ dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\sin bx)$ in integration]

$$= \frac{1}{\pi(n^2+1)}[e^{\pi}(0 - n\cos n\pi) - e^{-\pi}(0 - n\cos n\pi)]$$

$$= \frac{n(-1)^n(e^{-\pi} - e^{\pi})}{\pi(n^2+1)} = \frac{-2n(-1)^n \sinh\pi}{\pi(n^2+1)}$$

$$\therefore e^x = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\cos nx + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{n^2+1}\sin nx$$

$$= \frac{\sinh\pi}{\pi}\left[1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2+1} - \frac{n(-1)^n\sin nx}{n^2+1}\right].$$

Exercises.

1. Obtain the Fourier coefficient a_0 for the following functions.
   (i) $f(x) = x(2\pi - x)$ in $0 < x < 2\pi$
   (ii) $f(x) = |x|$ in $-\pi < x < \pi$
   (iii) $f(x) = x^2$ in $-\pi < x < \pi$
   (iv) $f(x) = x + x^2$ in $-\pi < x < \pi$
   (v) $f(x) = e^x$ in $-\pi < x < \pi$
   (vi) $f(x) = |\cos x|$ in $-\pi < x < \pi$
   (vii) $f(x) = \begin{cases} 0 & in\ -\pi < x < 0 \\ x & in\ \ 0 < x < \pi \end{cases}$
   (viii) $f(x) = \begin{cases} 2 & in\ \ 0 < x < \frac{2\pi}{3} \\ 1 & in\ \ \frac{2\pi}{3} < x < \frac{4\pi}{3} \\ 0 & in\ \ \frac{4\pi}{3} < x < 2\pi \end{cases}$
2. Obtain the Fourier coefficients b_n for the following functions.
   (i) $f(x) = x^2$ in $0 < x < 2\pi$
   (ii) $f(x) = \begin{cases} 0 & in\ \ 0 < x < \pi \\ 2\pi - x & in\ \ \pi < x < 2\pi \end{cases}$
   (iii) $f(x) = \sin(x/2)$ in $-\pi < x < \pi$
   (iv) $f(x) = \begin{cases} -\pi & in\ -\pi < x < 0 \\ x & in\ \ 0 < x < \pi \end{cases}$ Hence find b_3
3. Find the Fourier coefficients b_n for the following functions given below.
   (i) $f(x) = x^2$ in $0 < x < 2\pi$. Hence find b_2
   (ii) $f(x) = \begin{cases} -\pi & in\ -\pi < x < 0 \\ x & in\ \ 0 < x < \pi \end{cases}$ Hence find b_3
   (iii) $f(x) = \begin{cases} 0 & in\ -\pi < x \le 0 \\ x & in\ \ 0 < x \le \pi \end{cases}$ Hence find $\frac{b_1+b_2+b_3}{3}$
4. Obtain the Fourier series for the functions given below.

(i) $f(x) = \begin{cases} 1 & in \quad 0 < x < \pi \\ 2 & in \quad \pi < x < 2\pi \end{cases}$

(ii) $f(x) = \begin{cases} x & in -\pi \leq x \leq 0 \\ 0 & in \quad 0 \leq x \leq \pi \end{cases}$

(iii) $f(x) = \begin{cases} -x & in -\pi < x \leq 0 \\ 0 & in \quad 0 < x \leq \pi \end{cases}$

(iv) $f(x) = \begin{cases} -\pi & in -\pi < x < 0 \\ x & in \quad 0 < x < \pi \end{cases}$

5. If $f(x) = \begin{cases} \sin x & in \quad 0 \leq x \leq \pi \\ 0 & in \quad \pi \leq x \leq 2\pi \end{cases}$ obtain the Fourier series for $f(x)$ of periodicity 2π and hence evaluate $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots$
6. Find the Fourier series for $f(x) = \pi^2 - x^2$ in $-\pi < x < \pi$
7. Obtain the Fourier series for $f(x)$ given by $f(x) = \begin{cases} 1 + \frac{2x}{\pi} & in -\pi \leq x \leq 0 \\ 1 - \frac{2x}{\pi} & in \quad 0 \leq x \leq \pi \end{cases}$
8. Express $f(x) = (\pi - x)^2$ as a Fourier series in $0 < x < 2\pi$ and hence find the sum of the series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$
9. Find the Fourier coefficients a_n of the function $f(x) = \begin{cases} x & in \quad (0, \pi) \\ 2\pi - x & in \quad (\pi, 2\pi) \end{cases}$ with periodicity 2π

5.3 FOURIER SERIES - HALF RANGE

In several engineering and physical applications, it is required to obtainthe Fourier series expansion of a function in an interval $[0, l]$ where$l$ ishalf the period.Such an expansion is called**Half Range Fourier series.**

Half Range Sine Series.

Suppose$f(x)$ is defined in the interval$[0, l]$ We now define a newfunction as follows $(x) = \begin{cases} f(x) & if \quad 0 \leq x \leq l \\ -f(-x) & if \quad -l \leq x \leq 0 \end{cases}$. It is clear from the definition that $F(x)$ is an *odd function* defined in theinterval $[-l, l]$, Hence the Fourier series of $F(x)$contains only *sine* terms. Further in the interval $[0, l], F(x) = f(x)$and hence the *sine* series of $F(x)$gives the required *sine* series of $f(x)$ in the interval $[0, l]$. Thus

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \ where \ b_n = \frac{2}{l} \int_0^l f(x) \left(\frac{n\pi x}{l}\right) dx$$

Half Range Cosine Series.

We define$F(x) = \begin{cases} f(x) & if \quad 0 \le x \le l \\ f(-x) & if \quad -l \le x \le 0 \end{cases}$

Since $F(x) = f(-x). F(x)$ is an *even function*defined in the interval $[-l, l]$. Hence the Fourier series of $F(x)$ contains only *cosine* terms. Further in the interval $[0, l], F(x) = f(x)$ and hence the *cosine* series of $F(x)$ gives the cosine series of $f(x)$ in $[0, l]$.

Thus

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

Where,

$$a_0 = \frac{2}{l}\int_0^l f(x)\ dx \ and \ a_n = \frac{2}{l}\int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right)\ dx$$

Note: If $f(x)$ is defined in the interval $[0, \pi]$ then

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\ dx$$

Solved Problems.

Problem 1. Find the Fourier series for $f(x) = k$ in $0 < x < \pi$.

Solution. The Fourier *sine* series of $f(x)$ in the interval $0 < x < \pi$ is given by $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ where $b_n = \frac{2}{\pi}\int_0^l f(x) \sin nx\ dx$.

Now, $b_n = \frac{2}{\pi}\int_0^l k \sin nx\ dx = \frac{2k}{\pi}\left[\frac{\cos nx}{n}\right]_0^{\pi} = \frac{2k}{n\pi}[-\cos n\pi - (-1)]$

$$= \frac{2k}{n\pi}[1 - (-1)^n] = \begin{cases} \frac{4k}{n\pi} & if \quad n \ is \ odd \\ 0 & if \quad n \ is \ even \end{cases}$$

Hence the required sine series $f(x) = \sum_{n=1}^{\infty} \frac{4k}{(2n-1)\pi} \sin(2n-1)x$

$$= \frac{4k}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$$

Problem 2. Prove that the function $f(x) = x$ can be expanded

(i) In a series of cosines in $0 \le x \le \pi$ as

$$x = \frac{\pi}{2} - \frac{\pi}{4}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

(ii) In a series of sines in $0 \le x \le \pi$ as

$$x = 2\left[\frac{\sin x}{1} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \cdots\right]$$

Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}$

Solution. (i) $a_0 = \frac{2}{\pi}\int_0^\pi f(x)\ dx = \frac{2}{\pi}\int_0^\pi x\ dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^\pi = \pi$

$$Hence\ \frac{a_0}{2} = \frac{\pi}{2}$$

Now, $a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\ dx = \frac{2}{\pi}\int_0^\pi x\cos nx\ dx$

$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{[(-1)^n - 1]}{n^2}\right]$$

$$\therefore a_n = \begin{cases} -\frac{4}{\pi n^2} & if\quad n\ is\ odd \\ 0 & if\quad n\ is\ even \end{cases}$$

Hence the cosine series for $f(x) = x$ in $(0, \pi)$ is given by

$$x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\,x}{(2n-1)^2}$$

$$x = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

Putting $x = 0$ in the above result $x = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$

$$Hence, \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

(ii) Now, $b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\ dx = \frac{2}{\pi}\int_0^\pi x\sin nx\ dx$

$= \frac{2}{\pi}\left[-\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^\pi$

$$= \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right]$$

$$= \frac{-2(-1)^n}{n}$$

The sine series for $f(x) = x$ in $[0, \pi]$ is given as

$$Hence\ x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n} = 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right]$$

Putting $x = \frac{\pi}{2}$ in the above result we get

$$\frac{\pi}{2} = 2\left[\frac{\sin(\pi/2)}{1} - \frac{\sin\pi}{2} + \frac{\sin(3\pi/2)}{3} - \frac{\sin 2\pi}{4} + \cdots\right]$$

$$(ie)\frac{\pi}{2} = 2\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \cdots\right].$$

$$Hence\ 1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}.$$

Problem 3. Find the Fourier (i) cosine series and (ii) sine series for the function $f(x) = \pi - x$ in the imterval $(0, \pi)$.

Solution. (i) The Fourier cosine series of $f(x)$ is given by $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\ dx = \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\left(\frac{\pi^2}{2}\right) = \pi.$$

$$Now, a_n = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\cos nx\ dx$$

$$= \frac{2}{\pi}\left[(\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\cos nx}{n^2} + \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi n^2}[(-1)^{n+1} + 1]$$

$$= \begin{cases} \frac{4}{\pi n^2} & if\quad n\ is\ odd \\ 0 & if\quad n\ is\ even \end{cases}$$

Hence, $\pi - x = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$

$$x = \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

(i) The Fourier sine series of $f(x)$ is given by $\sum_{n=1}^{\infty} b_n \sin nx$ where b_n is given by

$$b_n = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\sin nx \; dx$$

$$= \frac{2}{\pi}\left[-(\pi - x)\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{n}\right] = \frac{2}{n}$$

Hence$\pi - x = 2\sum_{n=1}^{\infty} \frac{\sin nx}{n} = 2\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right]$

Problem 4. Find the half range cosine series for the function $f(x) = x^2$ in $0 \le x \le \pi$ and hence find the sum of the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

Solution. $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x) \; dx = \frac{2}{\pi}\int_0^{\pi} x^2 \; dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$

Hence $\frac{a_0}{2} = \frac{\pi^2}{3}$

$$Now, a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \; dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \; dx$$

$$= \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi\cos n\pi}{n^2}\right]$$

$$= \frac{4(-1)^n}{n^2}$$

The cosine series for $f(x) = x^2$ is given by

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx$$

$$= \frac{\pi^2}{3} + 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \cdots\right]$$

$$= \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right]$$

Put $x = 0$ in the above result we get

$$0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right]$$

$$Hence\ \left[\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\right]=\frac{\pi^2}{12}.$$

Problem 5. Obtain a cosine series for $f(x)=e^x$ in $0<x<\pi$.

Solution. $a_0=\frac{2}{\pi}\int_0^\pi f(x)\ dx=\frac{2}{\pi}\int_0^\pi e^x\ dx=\frac{2(e^\pi-1)}{\pi}$

Hence $\frac{a_0}{2}=\frac{(e^\pi-1)}{\pi}$

$$Now, a_n=\frac{2}{\pi}\int_0^\pi f(x)\cos nx\ dx$$

$$=\frac{2}{\pi}\int_0^\pi e^x\cos nx\ dx$$

$$=\frac{2}{\pi}\left[\frac{e^x(\cos nx+n\sin nx)}{1^2+n^2}\right]_0^\pi$$

$$=\frac{2}{\pi}\left[\frac{e^\pi\cos n\pi}{n^2+1}-\frac{1}{n^2+1}\right]$$

$$=\frac{2}{\pi}\left[\frac{e^\pi(-1)^n-1}{n^2+1}\right]$$

Hence the cosine series for $f(x)=e^x$ in $0<x<\pi$ is given by

$$e^x=\frac{e^\pi-1}{\pi}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[e^\pi(-1)^n-1]\cos\ nx}{n^2+1}$$

Problem 6. Find the Fourier sine series of the function $f(x)=\begin{cases} x & in \quad 0<x<\pi/2 \\ \pi-x & in \quad \pi/2<x<\pi \end{cases}$

Solution. The sine series for $f(x)$ is given by $f(x)=\sum_{n=1}^{\infty}b_n\sin nx$. Where

$$b_n=\frac{2}{\pi}\int_0^\pi f(x)\sin nx\ dx$$

Now, $b_n=\frac{2}{\pi}\int_0^{\pi/2}x\sin nx\ dx+\frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi-x)\sin nx\ dx$

$$=\frac{2}{\pi}\left[-\frac{x\cos nx}{n}+\frac{\sin nx}{n^2}\right]_0^{\frac{\pi}{2}}+\frac{2}{\pi}\left[-\frac{(\pi-x)\cos nx}{n}-\frac{\sin nx}{n^2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$=\frac{2}{\pi}\left[-\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right)+\frac{\sin(n\pi/2)}{n^2}\right]+\frac{2}{\pi}\left[\frac{\sin n\pi}{n^2}+\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right)+\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$=\frac{2}{\pi}\left[\frac{2}{n^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$= \frac{4}{\pi n^2} \sin\left(\frac{n\pi}{2}\right).$

Hence $f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi}{2}\right) \sin nx$

$$= \frac{4}{\pi}\left[\frac{\sin x}{1^2} + \frac{\sin \pi \sin x}{2^2} + \frac{\sin(3\pi/2)\sin 3x}{3^2} + \cdots\right]$$

$$= \frac{4}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right]$$

Problem 7. Find the half range sine series for $f(x) = x(\pi - x)$ in $(0, \pi)$. Deduce that

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots = \frac{\pi^3}{32}$$

Solution. The half range sine series for $f(x)$ in $(0, \pi)$ is given by

$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$. Where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \ dx$

Now, $b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x) \sin nx \ dx$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)(\pi - 2x) + \frac{\cos nx}{n^3}(-2)\right]_0^{\pi}$$

(by using Bernoulli's formula)

$$= \frac{2}{\pi}\left[(-\pi^2 + \pi^2)\frac{\cos nx}{n} - \frac{2\cos nx}{n^3} + \frac{2}{n^3}\right]$$

$$= \frac{4}{\pi n^3}(1 - \cos n\pi)$$

$$= \frac{4}{\pi}\left[\frac{1 - (-1)^n}{n^3}\right]$$

Hence $f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi}\left[\frac{1-(-1)^n}{n^3}\right] \sin nx$

$$= \frac{4}{\pi}\left[\frac{2\sin x}{1^3} + \frac{2\sin 3x}{3^3} + \frac{2\sin 5x}{5^3} + \cdots\right]$$

$$= \frac{8}{\pi}\left[\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots\right]$$

Put $x = \frac{\pi}{2}$ in the above result we get

$$f\left(\frac{\pi}{2}\right) = \frac{8}{\pi}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right]$$

$$(ie)\frac{\pi}{2}\left(\pi-\frac{\pi}{2}\right)=\frac{8}{\pi}\left[\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\cdots\right]$$

$$(ie)\frac{\pi^2}{4}=\frac{8}{\pi}\left[\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\cdots\right]$$

$$\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\cdots=\frac{\pi^3}{32}$$

Problem 8. Find the Fourier constant b_1 for the function $x\sin x$ in the half range $0<x<\pi$.

Solution.

Let $f(x)-x\sin x$. Then the half range Fourier series for $f(x)$ is given by $f(x)=\sum_{n=1}^{\infty} b_n \sin nx$. Where b_n is given by the formula

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\ dx$$

$$=\frac{2}{\pi}\int_0^{\pi}(x\sin x)\sin nx\ dx\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Put $n=1$ in (1) and we get

$$b_1=\frac{2}{\pi}\int_0^{\pi} x\ sin^2x\ dx$$

$$=\frac{2}{\pi}\int_0^{\pi}\frac{x(1-\cos 2x)}{2}\ dx$$

$$=\frac{1}{\pi}\left[\frac{x^2}{2}-\frac{x\sin 2x}{2}-\frac{\cos 2x}{4}\right]_0^{\pi}$$

$$=\frac{1}{\pi}\left(\frac{\pi^2}{2}-\frac{1}{4}+\frac{1}{4}\right)$$

Hence the Fourier constant $b_1=\frac{\pi}{2}$.

Exercise.

1. Obtain the half range sine series for the following functions

(i) $f(x)=x^2$ in $0<x<1$

(ii) $f(x)=e^x$ in $0<x<1$

(iii) $f(x)=x^3$ in $0<x<\pi$

(iv) $f(x)=\begin{cases}\frac{1}{4}-x & in 0<x<\frac{1}{2}\\ x-\frac{3}{4} & in\ \frac{1}{2}<x<1\end{cases}$

(v) $f(x)=\cos 2x$ in $0<x<\pi$

2. Obtain the half range cosine series for the following functions.

(i) $f(x) = \sin x$ in $0 < x < \pi$

(ii) $f(x) = \begin{cases} 0 & in \quad 0 < x < 1 \\ 1 & in \quad 1 < x < 2 \end{cases}$

(iii) $f(x) = \begin{cases} 1 & in \quad 0 < x < a/2 \\ -1 & in \quad a/2 < x < a \end{cases}$

5.4 FOURIER SERIES -ARBITRARY RANGE

So far, we have delt with Fourier series expansions having periods 2π or π. But in many of the problems the functions may have arbitrary periods(not necessarily2π). We now obtain Euler's formulae for Fouriercoefficients for functions having period $2l$ where$l$ is any *positive integer*.

Suppose $f(x)$ is defined in the interval $(-l, l)$.

Let $z = \frac{\pi x}{l}$.Hence$x = \frac{lz}{\pi}$. Also, when$x = -l$ wehave $z = -\pi$and when $x = l$ we have $z = \pi$Hence, the Fourier series of $F(z)$ is given by

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nz + b_n \sin nz).$$

Then we have $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(z)\ dz$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(z) \cos nz\ dz \ and\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(z) \sin nz\ dz$$

Hence $f\left(\frac{lz}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nz + b_n \sin nz)$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lz}{\pi}\right)\ dz$;

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lz}{\pi}\right) \cos nz\ dz \ and\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lz}{\pi}\right) \sin nz\ dz$$

We now go back to the original variable x by using the transformations $x = \frac{lz}{\pi}$ so that $dx = \frac{l}{\pi} dz$.

Thus

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]}$$

Where

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Note. The above formulae are valid for any interval of length $2l$.

Solved Problems.

Problem 1. If $f(x) = x$ is defined in the interval $-l < x < l$ with period $2l$. Find the Fourier expansion of$f(x)$.

Solution. Since $f(x) = x$ is an odd function $a_n = 0$ for all $n \geq 0$.

Now, $b_n = \frac{2}{l}\int_{-l}^{l} x\sin\left(\frac{n\pi x}{l}\right)dx$

$= \frac{2}{l}\left[-\frac{lx}{n\pi}\cos\left(\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right]_0^l$ (Bernoulli's formula)

$$= \frac{2}{l}\left(\frac{-l^2\cos n\pi}{n\pi}\right)$$

$$= 2\left(\frac{-l(-1)^n}{n\pi}\right)$$

$$= \frac{2(-1)^{n+1}l}{n\pi}$$

Hence the Fourier series is $x = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}l}{n}\sin\left(\frac{n\pi x}{l}\right)\right]$.

Problem 2. Find the Fourier series for $f(x) = x^2$ in $-1 < x < 1$.

Solution. Since $f(x)$ is an even function $b_n = 0$ for all n

Now, $a_0 = 2\int_0^1 f(x)\,dx = 2\int_0^1 x^2\,dx = 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}$

$$a_n = 2\int_0^1 x^2\cos n\pi x\,dx$$

$$= 2\left[\frac{x^2\sin n\pi x}{n\pi} + \frac{2x\cos n\pi x}{n^2\pi^2} - \frac{2\sin n\pi x}{n^3\pi^3}\right]_0^1$$

(Using Bernoulli's formula)

$$= 2\left(\frac{2\cos n\pi}{n^2\pi^2}\right)$$

$$= \frac{4(-1)^n}{n^2\pi^2}$$

Hence the Fourier series for $f(x)$ in $(-1,1)$ is given by

$$x^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n \cos n\pi x}{n^2}\right]$$

Problem 3. Find a Fourier sine series for $f(x) = ax + b$ in $0 < x < l$.

Solution. Since we have to find only the sine series of the Fourier series for the given function, we find only the Fourier coefficients b_n which is got from the formula

$$b_n = \frac{2}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_{-l}^{l}(ax+b)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[\frac{-(ax+b)l\cos\left(\frac{n\pi x}{l}\right)}{n\pi} + \frac{al^2\sin\left(\frac{n\pi x}{l}\right)}{n^2\pi^2}\right]_0^l$$

$$= \frac{2}{l}\left[\frac{-l(al+b)l\cos n\pi}{n\pi} + \frac{bl}{n\pi}\right] \text{ (justify)}$$

$$= \frac{2}{\pi}\left[\frac{b-(al+b)(-1)^n}{n}\right] \text{ (verify)}$$

Hence the sine series for $f(x)$ is given by

$$ax + b = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{b-(al+b)(-1)^n}{n}\right]\sin\left(\frac{n\pi x}{l}\right).$$

Problem 4. Find the half range Fourier sine series of $f(x) = x$ in $0 < x < 2$.

Solution. The half range Fourier sine series for $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{2}\right)$$

Where $b_n = \frac{2}{2}\int f(x)\sin\left(\frac{n\pi}{2}\right)$

$$= \int_0^2 \sin\left(\frac{n\pi}{2}\right) dx$$

$$= \left[\frac{-2\cos\left(\frac{n\pi x}{2}\right)}{n\pi} + \frac{4\sin\left(\frac{n\pi x}{2}\right)}{n^2\pi^2}\right]_0^2$$

$$= \left(\frac{-4\cos n\pi}{n\pi}\right)$$

$$= -\frac{4}{\pi}\left[\frac{(-1)^n}{n}\right]$$

Hence the Fourier series for $f(x)$ is given by

$$x = -\frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{2}\right)\right]$$

Exercises.

Find the Fourier series to represent the following functions:

1. $f(x) = \begin{cases} -1 & \text{in} \quad -2 < x < 0 \\ 1 & \text{in} \quad 0 < x < 2 \end{cases}$
2. $f(x) = x^2 - 2$ in $-2 < x < 2$
3. $f(x) = 2x - x^2$ in $0 < x < 3$
4. $f(x) = \begin{cases} 1 & \text{in} \quad -1 < x < 1 \\ 0 & \text{in} 1 < x < 3 \end{cases}$
5. $f(x) = (x-1)^2$ in $0 < x < 1$

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